Department of Physics and Astronomy
University of Heidelberg

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submitted by

Alexander Kunkel

born in Freital (Germany)

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Consistent Kaluza-Klein Dimensional Reduction of Variational Problems

This Bachelor Thesis has been carried out by Alexander Kunkel at the
Institute for Theoretical Physics in Heidelberg
under the supervision of
Prof. Dr. rer. nat. Johannes Walcher

Abstract

This thesis considers the consistency of Kaluza-Klein dimensional reductions. Giving necessary and sufficient conditions for consistent truncation, it aims to clarify the origin of the mathematical difficulties arising in dimensional reduction through symmetries on group manifolds and coset spaces. Furthermore, it presents examples of inconsistent truncation. This thesis is based on extensive desk research, including a thorough literature review. It begins by giving a definition for consistent truncation and then presents *Palais' Principle of Symmetric Criticality* which describes the conditions under which restricting a variational problem on a Banach manifold to the set of symmetric points under the action of a Lie group yields a consistent truncation. A proof of the *unimodularity condition* yields a stronger result for dimensional reduction on Lorentz manifolds. A brief study of the reduction of the gauge group in covariant theories shows the existence of a sector of Yang-Mills symmetries in the dimensionally reduced theory. The thesis concludes by proving that, in general, consistent truncation on coset spaces can only be achieved by discarding these gauge fields.

Zusammenfassung

Diese Arbeit untersucht die Konsistenz der Kaluza-Klein-Dimensionsreduktion von Variationsproblemen. Es werden notwendige und hinreichende Bedingungen für konsistente Dimensionsreduktionen gegeben und erklärt, welche Schwierigkeiten bei der Trunkierung von Variationsproblemen auf Gruppenmannigfaltigkeiten und Quotientenräumen auftreten. Außerdem werden Beispiele inkonsistenter Trunkierung vorgestellt. Diese Arbeit stützt sich auf eine umfangreiche Literaturrecherche. Nach einer Definition von *konsistenter Trunkierung* wird Palais' Prinzip vorgestellt, welches Kriterien liefert, wann die Einschränkung eines Variationsprinzips auf die Menge der symmetrischen Punkte unter der Wirkung einer Lie-Gruppe auf einer Banach-Mannigfaltigkeit eine konsistente Trunkierung liefert. Die anschließend bewiesene *Unimodularitäts-Bedingung* stellt ein stärkeres Kriterium für konsistente Dimensionsreduktion auf Lorentz-Mannigfaltigkeiten dar. Die Untersuchung der Reduktion der Eichgruppe einer kovarianten Theorie zeigt die Existenz eines Sektors von Yang-Mills-Symmetrien in der niedrigdimensionalen Theorie auf. Zum Schluss wird dargelegt, dass konsistente Trunkierung bei Dimensionsreduktion durch Quotientenräume im Allgemeinen nur erreicht werden kann, indem die oben genannten Yang-Mills-Eichfelder zu Null gesetzt werden.

Contents

Notations and conventions

Chapter 2 & 3 & 4.2

X	arbitrary set
$\mathscr{M}$	(infinite-dimensional)-Banach manifold
M_G	n-dimensional manifold with isometry group G
G	n-dimensional Lie group
Σ	set of symmetric points under group action

- lower-case Greek indices $\mu, \nu, \eta ... = 1, ..., d$
- lower-case Latin indices $i, j, k ... = d + 1, ..., n$
- upper-case Latin indices $A, B, C ... = 1, ..., d + n$

Chapter 4.1

G	Lie group
$H \subseteq G$	Lie subgroup of G
$\mathfrak{g}$	Lie algebra of G
$\mathfrak{h}$	Lie algebra of H

- coset space indices are denoted with lower-case Greek alphabets
- tangent space indices with Latin alphabets
- lower-case Greek indices $\mu, \nu, \eta =$ dim $\mathfrak{h}$ +1, ..., dim $\mathfrak{g}$
- lower-case Latin indices $i, j, k =$ 1, ... dim $\mathfrak{h}$
- lower-case Latin indices $a, b, c =$ dim $\mathfrak{h}$ +1, ..., dim $\mathfrak{g}$
- upper-case Latin indices $A, B, C =$ 1, ..., dim $\mathfrak{g}$

Chapter 5

$\mathscr{M}$	d-dimensional Lorentz manifold
G/H	n-dimensional coset space

- lower-case Greek indices $\mu, \nu, \eta ... = 1, ..., d$
- lower-case Latin indices $i, j, k ... = d + 1, ..., n$
- upper-case Latin indices $A, B, C ... = 1, ..., d + n$

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Introduction

In physics, one often wants to use symmetry information prior to implementing a variational principle in order to simplify the computation of the resulting equations of motion (e.o.m.). For instance, one wants to reduce the number and simplify the form of the resulting field equations. Given a variational principle which is invariant under the action of a symmetry group, one usually checks whether a symmetric field configuration is an extremum of the action by verifying whether the first variation of the action into directions that are also symmetric vanishes. Suppose one is interested in spherically symmetric solutions of the Laplace equation in $\mathbb{R}^3$. They are invariant under the standard action of $SO(3)$ on $\mathbb{R}^3$. On the one hand, one can choose the ansatz $\phi(x, y, z) = q(r)$ that respects the spherical symmetry and the Laplace equation simplifies to

$$q'' + \frac{2}{r} q' = 0 \ . \tag{1}$$

On the other hand, the Laplace equation can also be obtained by varying the action

$$S = \frac{1}{2} \int ||\nabla \phi||^2 \, \mathrm{d}V \ . \tag{2}$$

Substituting the rotationally invariant ansatz for ϕ into this action and performing the variation yields (1). Whether one enforces the symmetry condition at the level of the variational principle or at the level of the e.o.m. does not matter. Note that the above procedure can be understood as *dimensional reduction*. By enforcing spherical symmetry, one reduces the theory from three dimensions to one dimension.

One can apply this approach to an arbitrary field theory by demanding that the theory admit a symmetry group. The first step is to make a group-invariant ansatz for the fields in the theory. This ansatz is then substituted either into the original Lagrangian or the higher-dimensional e.o.m. derived from it. The process of modifying the variational principle in such a way is called *truncation* or *reduction*. One could also consider the reduction of a field theory through constraints that would reduce the number of independent fields defining the theory. However, this thesis shall focus on reduction through symmetry groups where the dimension of the space-time underlying the field theory is reduced. This type of reduction is also called *Kaluza-Klein dimensional reduction*. The main aim of this thesis is to understand the conditions for consistent Kaluza-Klein dimensional reduction.

A reduction is called *consistent* if the implementation of the truncation at the level of the variational principle agrees with the implementation of the truncation at the level of the e.o.m. (Pons & Talavera, 2003). One can describe consistent truncation graphically by demanding that the following diagram be commutative:

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\quad Reduction \quad} & \mathcal{L}_{Red} \\ \big\downarrow {\scriptstyle Variation} & & \big\downarrow {\scriptstyle Variation} \\ \frac{\delta\mathcal{L}}{\delta\Phi} = 0 \xrightarrow{Reduction} (\frac{\delta\mathcal{L}}{\delta\Phi})_{Red} = 0 & \Longleftrightarrow & \frac{\delta\mathcal{L}_{Red}}{\delta\Phi} = 0 \ . \end{array}$$

If a truncation is inconsistent, solutions of the e.o.m. for the reduced Lagrangian $\mathcal{L}_{Red}$ may not be solutions of the e.o.m. for the original Lagrangian $\mathcal{L}$. A consistent truncation in the above sense guarantees that field configurations that are extrema of the reduced action are also extrema of the original action. One can easily see that a truncation is not always consistent in the above sense. For instance, if one substitutes too much information about the structure of the solutions back into the action integral, the variation may fail. For example, consider the action

$$S[x(t)] = \int_{t_0}^{t_f} \frac{1}{2} m\dot{x}(t)^2 \mathrm{d}t \ . \tag{3}$$

Applying the Euler-Lagrange equations, one obtains $\ddot{x}(t) = 0 \Rightarrow x = A \cdot t + B;\ A, B \in \mathbb{R}$. Substitution of the e.o.m. $x = A \cdot t + B$ back into S and application of the Euler-Lagrange equation yield

$$S = \int_{t_0}^{t_f} \frac{1}{2} mA^2 \mathrm{d}t = const. \implies \delta S = 0 \ . \tag{4}$$

These are not the e.o.m. obtained before and the truncation is therefore inconsistent.

Responding to the unresolved issue of inconsistent truncation, this thesis aims to make two contributions. In the first part, the thesis aims to understand the conditions for consistent Kaluza-Klein reduction on group manifolds.

One of the first publications in physics recognising the issue of inconsistency was Hawking (1969). Hawking was working on homogeneous Bianchi cosmology which considers pure 4-dimensional general relativity dimensionally reduced via a 3-dimensional Lie group. He realised that substitution of the group invariant-ansatz into the Hilbert-Einstein action will not always yield the correct field equations if the trace of the structure constants of the Lie group had some non-vanishing component. Sneddon (1976) clarified that the truncation is consistent and one obtains the correct e.o.m. if the symmetry group is unimodular. Scherk and Schwarz (1979) derived in a more general framework that the tracelessness of the structure constants is indeed a necessary condition for the consistency of dimensional reductions. It thus turns out that the issue of consistency can be decided purely based on properties of the symmetry group.

In the mathematical literature, consistent truncations have been studied by Palais (1979) who coined the term *Principle of Symmetric Criticality*. Given a group action on a space of fields, one can consider the restriction of an action functional S to the group invariant fields to obtain the reduced action $\hat{S}$. Palais showed that given a compact Lie group, the reduced action $\hat{S}$ will always yield the reduced field equations.

In the second part, the thesis aims to clarify another type of difficulty arising when implementing dimensional reductions on coset spaces. As was to be expected after what Palais had shown, one can obtain consistent truncation on compact coset spaces such as spheres. However, in doing so, one sacrifices the sector of Yang-Mills symmetries that one would otherwise obtain in the lower-dimensional theory. The goal is then to make an ansatz for the fields in the theory that is *not invariant* under the action of the symmetry group. In general, one will not obtain consistent truncation in this case and can only hope for consistent truncation in a weaker sense: One starts with a Lagrangian, introduces some ansatz for the reduction of the fields which is plugged into the higher-dimensional e.o.m.. If the dependence on the variables of the higher dimensional space cancels out, the original e.o.m. are compatible with such an ansatz and the reduced e.o.m. will be considered a consistent truncation of the former ones (Cvetic, Gibbons, Lu & Pope, 2003).

The rest of this thesis is structured as follows: Chapter 2 explains Palais' principle and offers the main ingredients for the proof of consistent reduction through a compact symmetry group. Moreover, it shows that group truncation may fail at the example of an action of $\mathbb{R}$.

Chapter 3 provides a proof of the *unimodularity condition*. It states that dimensional reduction induced by a set of independent Killing vectors generating a unimodular group on a Lorentz manifold is necessarily consistent. This proof provides a different perspective on Palais' principle and highlights which field components survive the truncation. A brief study of the reduction of the gauge group in covariant theories then demonstrates the existence of a sector of Yang-Mills symmetries in the lower-dimensional theory.

Chapter 4 extends the proof of chapter 3 to the case of reductions on coset spaces. It is shown that, in general, consistent truncation can only be achieved by discarding the Yang-Mills gauge bosons.

Chapter 5 concludes by pointing out the inconsistency of general coset space reductions through the derivation of the dimensionally reduced e.o.m. from the Hilbert-Einstein Lagrangian. It uses an ansatz for the metric which keeps the Yang-Mills gauge bosons and is therefore not group-invariant.

2 Symmetric Criticality

This chapter presents Palais' *Principle of Symmetric Criticality* (1979), henceforth also referred to as *Principle.* The Principle guarantees consistent truncation through restriction of a variational principle to the set of symmetric points Σ under the action of suitable groups on Banach-manifolds. This approach is very useful for understanding the issues arising in coset space reductions. This chapter heavily draws on Palais (1979) and presents necessary conditions for the set of symmetric points Σ to be a smooth submanifold of $\mathscr{M}$ and the principle to hold. Before the Principle is discussed in detail, a few definitions are introduced:

Definition 1: Left group action of G on X

Let G be a group and let X be a set.
A map $\mu_L : G \times X \to X$ is called *left action* of G if

- $\mu_L(e, x) = x \qquad \forall x \in X$ where e is the identity element of G
- $\mu_L(gh, x) = \mu_L(g, \mu_L(h, x)) \qquad \forall g, h \in$G $\forall x \in X$

The *right action* of G is defined analogously.

The *left translation* is defined as $L_g(x) := \mu_L(g, x) =: gx$. Analogously, $R_g(x) := \mu_R(x, g) =: xg$. Then $G \cdot x := \{g \cdot x | g \in G\}$ for $x \in X$ is called the orbit of x. The group action is *transitive* if there exists $x \in X$ such that $\mathbf{G} \cdot x = X$.

Definition 2: G-manifold

A smooth manifold $\mathscr{M}$ is called a G-manifold if it is endowed with a left group action for a group G and if $\forall g \in G$ $L_g(x)$ is a smooth map on $\mathscr{M}$. If G is a Lie group, the map $\mu_L : G \times \mathscr{M} \to \mathscr{M}$ further is required to be smooth in both arguments.

Let $\mathscr{M}$ be a smooth G-manifold and let $f : \mathscr{M} \to \mathbb{R}$ be a smooth G-invariant function on $\mathscr{M}$. $p \in \mathscr{M}$ is called a *critical point of* $\mathscr{M}$ of f if

$$df_p(X) = 0 \qquad \forall X \in T_p\mathscr{M} \ . \tag{5}$$

$p \in \mathscr{M}$ is called a *symmetric point of* $\mathscr{M}$ if

$$p \in \Sigma := \{p \in \mathscr{M} | g \cdot p = p \; \forall g \in G\} \ . \tag{6}$$

The Principle now describes the conjecture that a symmetric point p that is a critical point of $f|_\Sigma$ is also a critical point of f:

Conjecture 1: Palais' *Principle of Symmetric Criticality*

$$df_p(X) = 0 \; \forall X \in T_p\Sigma \Rightarrow df_p(X) = 0 \; \forall X \in T_p\mathscr{M} \tag{7}$$

In general, the Principle is neither well-defined when Σ is not a smooth submanifold of $\mathscr{M}$, nor valid for arbitrary groups G. $\mathscr{M}$ could in general be a section of a smooth fibre bundle, e.g. the field space of a physical theory and f the action functional. Σ would then be the set of G-invariant fields. Palais principle states that the vanishing of the variation of f in directions tangential to Σ implies that the variation of f in directions transverse to Σ also vanishes. That is, fields that are extrema of the truncated action functional $f_R := f|_\Sigma$ are also extrema of the original higher dimensional action. Writing $i : \Sigma \to \mathscr{M}$ for the embedding of Σ into $\mathscr{M}$, one can recast the principle as

$$d(i^* f)|_p = 0 \iff df|_p = 0. \tag{8}$$

Since the exterior derivative and the pullback commute, it is evident that the Principle is equivalent to consistency in the sense mentioned in the introduction: the implementation of the truncation at the level of the Lagrangian and at the level of the e.o.m. commute.

2.1 Linearisability

Let p be a symmetric point of a smooth G-manifold $\mathscr{M}$. The representation of G in $T_p\mathscr{M}$ given by $\rho : G \to T_p\mathscr{M};\, g \mapsto D_p g$ is called the *linearisation of the action of G at p*. Further, G is called *linearisable about p* if there exists a coordinate system (ϕ, U) of $\mathscr{M}$ such that for each $g \in G$ there exists a linear map $\tilde{g} : V \to V$ with $\tilde{g}|_{\phi(U)} = \phi \circ g \circ \phi^{-1} : \phi(U) \to V$.

Proposition 1: Σ smooth submanifold

If $\mathscr{M}$ is a smooth G-manifold such that the action of G is linearisable about each symmetric point, then the set Σ of symmetric points is a smooth submanifold of $\mathscr{M}$.

Proof.

$$\mathbf{W} = \{v \in V | \tilde{g}v = v \; \forall g \in G\} \tag{9}$$

is a closed linear subspace of V since the map $\tilde{g}$ is linear and one has:

$$\mathbf{W} \cap \phi(U) = \{v \in V \cap \phi(U) \,|\, (\phi \circ g \circ \phi^{-1})\, v = v \; \forall g \in G\} \; . \tag{10}$$

Let $v \in \mathbf{W} \cap \phi(U)$. Then $\phi^{-1}(v) = \phi^{-1} \circ \phi \circ g \circ \phi^{-1}(v) = g \circ \phi^{-1}(v) \implies \phi^{-1}(v) \in U \cap \Sigma$. Now clearly, $\phi|_\Sigma : U \cap \Sigma \to \phi(U) \cap \mathbf{W}$ is a diffeomorphism. Therefore, Σ is locally a smooth submanifold of $\mathscr{M}$ at p. □

2.2 Palais-condition

Let V be a Banach G-space, that is a Banach space with a linear representation of G. The set Σ of symmetric points in V is a linear subspace of V. Let V^* be its dual space with the natural linear action of G defined as

$$(gl)(v) := l(g^{-1}v) \tag{11}$$

for $l \in V^*$ and $g \in G$. Let Σ_* denote the set of symmetric points in V^*. It follows that

$$\begin{aligned} \Sigma_* &= \{l \in V^* \,|\, gl = l\} \\ &= \{l \in V^* \,|\, gl(v) = l(g^{-1}v) = l(v) \; \forall v \in V \; \forall g \in G\} \\ &= \{l \in V^* \,|\, l(gv) = l(v) \, \forall v \in V \; \forall g \in G\} \\ &= \{\text{set of } G\text{-invariant functionals on } V\} \; . \end{aligned} \tag{12}$$

Let $W \subseteq V$ be a linear subspace of V and $W^0 \subseteq V^*$ the corresponding annihilator subspace. Consider a linear map $l : V \to \mathbb{R}$. Its differential is a linear map $dl_p : T_pV = V \to T_{l(p)}\mathbb{R} = \mathbb{R}$ which fulfills $l = dl_p \; \forall p \in V$ since l is linear. If p is a critical point in W, then

$$d(l|_W)_p(v) = l|_W(v) = 0 \; \forall v \in W \iff l \in W^0$$

. It follows that all points in W are critical.

Proposition 2: Palais condition

If V is a Banach G-space, then the *Palais-Condition*

$$\Sigma_* \cap \Sigma^0 = 0$$

is both a necessary and sufficient condition for the *Principle of Symmetric Criticality* to be valid for smooth G-invariant functions $f : V \to \mathbb{R}$.

Proof. Let $p \in \Sigma$ be a critical point of $f|_\Sigma$ and define $l := df_p$. Then, $l|_\Sigma = df_p|_\Sigma = d(f|_\Sigma)_p = 0$ and therefore $l \in \Sigma^0$.
"$\Rightarrow$":

Let the Palais-Condition hold true. One needs to show $l = 0$. Since f is G-invariant, $f = f \circ g \; \forall g \in G$ and therefore $df_v = d(f \circ g)_v = d(g^* f)_v = (g^* df)_v = df_{gv} \circ g \; \forall v \in V$. Taking $v = p \in \Sigma$ yields $g^{-1} l = l \; \forall g \in G$ and therefore $l \in \Sigma_*$. Hence, $l \in \Sigma_* \cap \Sigma^0 = 0 \implies l = df_p = 0$.
"$\Leftarrow$":

Let the Principle of Symmetric Criticality hold true and let $l \in \Sigma_* \cap \Sigma^0$. According to our foregoing consideration, all points in Σ are critical points of $f|_\Sigma$. □

Definition 3: Admissible space

A Banach G-space V is called *admissible* if it satisfies the *Palais-Condition.* A smooth G-manifold $\mathscr{M}$ is called admissible if for all symmetric points $p \in \mathscr{M}$ the action of G on $\mathscr{M}$ is linearisable about p and if the linearisation of G at p is an admissible Banach G-space.

Theorem 1: Palais theorem for admissible manifolds

The Principle of Symmetric criticality is valid for admissible smooth G-manifolds.

Proof. The action of G is linearisable about each symmetric point p. Further, one can check in a sufficiently small neighborhood of p whether p is a critical point of f and the theorem follows from proposition (2). □

We shall not provide the proof of the Principle for compact Lie groups here. For a proof refer to Palais (1979, p. 28ff). Palais goes on to show that every Banach G-space for a compact Lie group G is admissible, that the C^1-action of such a G on a Banach manifold can be linearised about any symmetric point and concludes:

Theorem 2: Palais' principle for compact groups

If G is a compact Lie group, then any smooth G-manifold $\mathscr{M}$ is admissible and hence the *Principle of Symmetric Criticality* is valid for $\mathscr{M}$.

Palais also showed that the principle is valid for a connected semi-simple Lie group acting real analytically on a finite dimensional real analytic G-manifold and for G acting isometrically on Riemannian manifolds. We shall see the following chapters in view of theorem (2). Especially, for the interesting case of Kaluza-Klein-reduction on spheres, it guarantees consistent truncation because the left isometry group of $SO(n+1)/SO(n) = \mathbb{S}^n$ is $SO(n+1)$ and therefore compact.

2.3 Counterexample: Flow on $\mathbb{R}^n$

This section shows that the Principle may fail to hold true, even when well-defined. Consider the action of $\mathbb{R}$ on $\mathscr{M} = \mathbb{R}^n$ defined via the vector field:

$$\mathscr{X} = \prod_{i=1}^{n-1} x_i \partial x_n \ . \tag{13}$$

If $\psi : \mathbb{R} \to \mathbb{R}^n$ is the solution of the initial value problem:

$$\dot{\psi}(t) = \mathscr{X}(\psi(t)), \qquad \psi(0) = \psi_0 = (\psi_1, \psi_2, ..., \psi_n) \ . \tag{14}$$

$\gamma_t(\psi_0) = \psi(t)$ is the flow of the vector field $\mathscr{X}$. For the given vector field, the globally defined flow is given as

$$\gamma_t(\mathbf{x}) = (x_1, x_2, ..., x_{n-1}, t \cdot \prod_{i=1}^{n-1} x_i + x_n) \ . \tag{15}$$

The set of symmetric points Σ is the x_n-axis, hence a smooth submanifold of $\mathbb{R}^n$. This also follows from proposition (1) since this action of $\mathbb{R}$ is linear about all points in $\mathbb{R}^n$. A smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ is invariant under this action of $\mathbb{R}$ if

$$\phi(x_1, x_2, ..., x_{n-1}, x_n) = \phi(x_1, x_2, ..., x_{n-1}). \tag{16}$$

holds true. Hence,

$$\begin{aligned}\Sigma_* &= \{l : \mathbb{R}^n \to \mathbb{R} \mid l \text{ linear and } l(x_1, x_2, ..., x_n) = l(x_1, x_2, ..., x_{n-1})\} \\ &= \Sigma^0\end{aligned} \tag{17}$$

implies that $\Sigma_* \cap \Sigma^0 \neq 0$. Therefore proposition (2) states that the Principle will fail in this setting. For instance, consider the smooth $\mathbb{R}^n$-function

$$\phi(\mathbf{x}) = \prod_{i=1}^{n-1} x_i \ . \tag{18}$$

It is invariant under $\mathscr{X}$ and

$$d(\phi|_\Sigma) = \partial x_n(0(x)) = 0 \ \forall x \in \Sigma \ . \tag{19}$$

Thus, all symmetric points in Σ are also critical points of ϕ. Yet, ϕ does not have any critical points in $\mathbb{R}^n$.

3

Independent Killing vectors

As opposed to the general Banach manifold case considered in chapter 2, this chapter looks at consistent truncation in the context of diffeomorphism-invariant field theories on Lorentz manifolds. The set of symmetric points Σ introduced in chapter 2 is characterised through the Killing condition. Furthermore, the *tracelessness condition* for consistent truncation is proved in the case that the symmetry group is generated by linearly independent Killing vectors.

Definition 4: One-parameter group of diffeomorphisms ϕ_t

Consider a smooth map $f : \mathbb{R} \times \mathscr{M} \to \mathscr{M}$ such that $\phi_t : \mathscr{M} \to \mathscr{M}$, $\phi_t(p) := f(t,p)$ is a diffeomorphism $\forall t \in \mathbb{R}$ and $\phi_t \circ \phi_s = \phi_{t+s} \, \forall t, s \in \mathbb{R}$.

Let $p \in \mathscr{M}$, then $\phi_t(p) : \mathbb{R} \to \mathscr{M}$ is a curve. One can define the value of v at p as $v_p := \frac{d\phi_t(p)}{dt}|_{t=0}$ and obtains a smooth vector field. Hence, associated to a one-parameter group of finite transformations of $\mathscr{M}$ there is a vector field that is the infinitesimal generator of these transformations.

Definition 5: Isometry

Let (M, g) be a pseudo-Riemannian manifold and consider a coordinate transformation that leaves the form of the metric invariant: $g'_{ab}(x) = g_{ab}(x)$. Such a transformation is called an *isometry* of the metric.

An infinitesimal isometry is described by a vector ψ called *Killing vector*, which is said to *generate* the isometry. A Killing vector satisfies

$$\begin{aligned} &\mathcal{L}_\psi g = 0 \\ \implies &\mathcal{L}_\xi g_{ab} = \partial_a \xi^c g_{cb} + \partial_b \xi^c g_{ac} + \xi^c \partial_c g_{ab} = 0 \; . \end{aligned} \tag{20}$$

One can replace ∂ by ∇ since the additional terms cancel because of the anti-symmetry of the Lie derivative. Using $\nabla_c g_{ab} = 0$, one deduces

$$\nabla_a E_b + \nabla_b E_a = 0 \tag{21}$$

which is known as *Killing equation.* The set of isometries of a manifold $\mathscr{M}$ has the structure of a Lie group and is called the *symmetry group of M.* The symmetry group of an m-dimensional manifold has maximum dimension $\frac{m(m+1)}{2}$. Isometries are obtained from the Killing vectors by exponentiation in the same way that group elements are obtained from the infinitesimal generators which form the Lie algebra of the group. Let G be a Lie group with structure constants C^s_{ij}. A manifold $\mathscr{M}$ is said to be *invariant under the group* G if there are dim $G = n$ Killing vector fields which obey the Lie algebra relation

$$[\xi_i, \xi_j] = C^s_{ij}\xi_s \ . \tag{22}$$

A Lie group G is called *simply transitive on subspaces* if the ξ_i are linearly independent as vector fields, that is if

$$\sum_i a_i \psi_i = 0 \implies a_i = 0. \tag{23}$$

where a_i are smooth functions.

Definition 6: Left/right invariance

Let L_g/R_g be the left/right translation of a Lie group G acting on G, let L^*_g/R^*_g be the pushforward of the left/right translation. A vector field X on G is *left-invariant* (under the action of G) if it satisfies

$$L^*_g X_h = X_{gh} \ \forall h \in G \ .$$

Analogously, a vector field X on G is *right-invariant* if it satisfies

$$R^*_g X_h = X_{hg} \ \forall h \in G \ .$$

More generally, a tensor field T on G left-invariant if $L^*_g T = T$ and right-invariant if $R^*_g T = T$.

Let $(\mathscr{M}, g)$ be a d-dimensional pseudo-Riemannian manifold with symmetry group G and let ξ_i be its Killing vectors. It is useful to introduce a left-invariant tangent-space basis. The members of such a basis are left-invariant vector fields X_a $(a, b, c = 1, ..., d)$. They have zero Lie derivative with respect to the Killing vectors ξ_i $(i = 1, ..., \dim G)$:

$$\mathcal{L}_{\xi_i}(X_a) = [\xi_i, X_a] = 0 \; . \tag{24}$$

Moreover, consider the Lie derivative of $g(X_a, X_b)$ with respect to the Killing vectors ξ_i:

$$\begin{aligned} \xi_i(g(X_a, X_b)) &= \mathcal{L}_\xi g(X_a, X_b) \\ &= (\mathcal{L}_\xi g)(X_a, X_b) + g(\mathcal{L}_\xi X_a, X_b) + g(X_a, \mathcal{L}_\xi X_b) \\ &= g([\xi, X_a], X_b) + g(X_a, [\xi, X_b]) \end{aligned} \tag{25}$$

(24) shows that the components of the metric $g_{\mu\nu}$ in a left-invariant basis satisfy

$$\mathcal{L}_\xi g_{\mu\nu} = \mathcal{L}_\xi g(X_\mu, X_\nu) = 0 \; . \tag{26}$$

A left-invariant vector field X_h is determined by its value at the tangent space of the identity element since $X_h = L_h^*(X_e)$. Thus, if one chooses an arbitrary element v at the tangent space of the identity of G, it can be extended to a left-invariant vector field on G via $X_g := L_g^*(v)\ \forall g \in G$. One can easily show that the flow generated by the left-invariant vector field X_g is the one-parameter group of right translations

$$\psi_t(g) = g \exp(tX_g) \; . \tag{27}$$

Hence, the *left-invariant vector fields are infinitesimal generators of right translations.* The same holds true for right-invariant vector fields and left translations. This shows that a right-invariant field Y is automatically invariant under the flow generated by a left-invariant field X. Their bracket thus vanishes:

$$[X, Y] = 0 \; . \tag{28}$$

Definition 7: Homogeneous space

A homogeneous space is a smooth manifold $\mathscr{M}$ endowed with the smooth, transitive action of a Lie group.

Let $\mathscr{M} \times M_G$ be a $(d+n)$-dimensional space-time manifold with a Lorentzian metric tensor $\mathbf{g_{AB}}$ $(A, B, C = 1, ..., d+n)$ and let G be the n-parameter isometry group of M_G. Let $\mathscr{N}$ be the field space of a physical theory on $\mathscr{M}$ such as the tensor bundle over $\mathscr{M}$. The action functional S is a smooth map $S : \mathscr{N} \to \mathbb{R}$ and the set of symmetric points is given as before as $\Sigma = \{X \in \mathscr{N} | gX = X \ \forall \ g \in g\}$. Let ξ_i be the Killing vectors generating G. An arbitrary field $X \in \mathscr{N}$ is invariant under the action of G if

$$\begin{aligned} &\mathcal{L}_{\xi^a} X = 0 \\ \Longrightarrow \ &\Sigma = \{X \in \mathscr{N} | L_{\xi_a} X = 0 \ \forall a\} \ . \end{aligned} \tag{29}$$

The truncation of the variational principle is implemented by requiring the fields in the Lagrangian to fulfill (29). This is a truncation to the singlets under the group action just as in the Palais case:

$$g \in \Sigma \iff \mathcal{L}_{\xi_a} g = 0 \ . \tag{30}$$

Using a different approach then in chapter 2, it is shown that this truncation is consistent if the isometry group G is *unimodular*. This is equivalent to the the trace of the structure constants C^c_{ab} of the Lie algebra $\mathfrak{g}$ of the isometry group vanishing:

$$C^c_{ac} = 0 \qquad \forall a \in \{1, ..., n\} \ . \tag{31}$$

Compact groups are an example of unimodular groups. Therefore, this case is already covered by Palais' theorem. However, the following treatment, closely following Pons and Talavera (2003, p. 5ff), is very instructive since the implementation of the truncation is explicitly carried out. The field components surviving the truncation are explicitly characterised. This will turn out to be essential for an understanding of coset space reductions.

3.1 Unimodularity condition

The Killing vectors $\xi_\mathbf{a}$ are the infinitesimal generators of left translations L^*_g on $\mathscr{M}$ and are thus right-invariant. They span a Lie algebra $\mathfrak{g}$ with structure constants C^c_{ab}:

$$[\xi_a, \xi_b] = C^c_{ab} \xi_c \ . \tag{32}$$

The group of isometries G yields a foliation of $\mathscr{M}$ into n-dimensional, space-like (all of the tangent vectors to that surface are everywhere space-like), homogeneous surfaces. Every leaf of the foliation $\mathscr{N}$ is diffeomorphic to G and supports its own realisation of the Lie algebra $\mathfrak{g}$. This section treats them as copies of G. Next, local coordinates on $\mathscr{M}$ are introduced such that y^μ are longitudinal to the foliation surfaces and x^μ are transverse to the foliation surfaces. The coordinates

x^μ will survive the truncation whereas y^μ will vanish. In these local coordinates, the Killing vectors ξ_a take the general form

$$\xi_a = \xi_a^\alpha(x, y)\partial_{y^\alpha} \ . \tag{33}$$

One obtains a basis $(Y_1)_e, ..., (Y_n)_e \in T_e\mathscr{N}$ of the tangent space at the identity of each foliation surface via $(Y_i)_e := R_{g^{-1}}(\xi_a)_g$ and extends it to a set of n linearly-independent, left-invariant vector fields $(Y_1)_g, ..., (Y_n)_g \in T_g\mathscr{N}$ via $(Y_i)_g := L_g^*(Y_i)_e$. Since L_g is a diffeomorphism, L_g^* is a tangent space isomorphism and $L_g^*X_1, ..., L_g^*X_n$ constitute a basis of $T_g\mathscr{N}$. By definition $(Y_i)_e = (\xi_i)_e$, one obtains:

$$[\mathbf{Y_a}, \mathbf{Y_b}] = -C_{ab}^c\mathbf{Y_c} \ . \tag{34}$$

One can now express the vector fields $\mathbf{Y_a}$ in the local basis as

$$\mathbf{Y}_a = y_a(x, y)^\alpha\partial_{y^\alpha} \ . \tag{35}$$

Since left- and right-invariant vector fields commute, it follows that $\mathcal{L}_{\xi_a}\mathbf{Y}_b = [\xi_a, \mathbf{Y}_b] = 0$. Furthermore, note that according to the previous section Y_b generate right translations on G because they are left-invariant. On every surface of foliation, a basis of 1-forms ω^a is defined via

$$\omega^a = \omega_\alpha^a(x, y)dy^\alpha. \tag{36}$$

such that $\omega^a \cdot \mathbf{Y}_b = \omega_\alpha^a \cdot Y_b^\alpha = \delta_b^a$. Note that this basis is automatically left-invariant since

$$\begin{aligned} L_g^*\omega_g(Y_e) &= \omega_g(L_{g^*}Y_e) \\ &= \omega_g(Y_g) = \omega_e(Y_e) \\ &\implies L_g^*\omega_g = \omega_e \\ &\implies \mathcal{L}_{\xi_a}\omega^b = 0 \ . \end{aligned} \tag{37}$$

This will turn out to be the important difference with the coset space reduction case. Expressing the metric using the mixed basis of 1-forms $\{dx^\mu, \omega^a\}$ yields

$$g = g_{\mu\nu}(x, y)dx^\mu dx^\nu + g_{ab}(x, y)(A_\mu^a(x, y)dx^\mu + \omega^a)(A_\mu^b(x, y)dx^\nu + \omega^b) \ . \tag{38}$$

In the next step, one needs to implement the Killing condition on the metric:

$$\mathcal{L}_{\xi_a}g = 0 \ . \tag{39}$$

In other words, one imposes conditions on the metric components such that the vectors ξ_a are Killing vectors of the metric. As argued earlier, this is the desired truncation of the metric. Note that one has to reduce all fields in a theory to dimensionally reduce it. Let ∂_{x^η} be the basis vectors of the lower dimensional tangent space dual to dx^η:

$$\begin{aligned}
&\mathcal{L}_{\xi_a}(g_{\mu\nu}dx^\mu dx^\nu)(\partial_{x^\eta},\partial_{x^\psi}) \\
&\quad = \xi_a(g_{\eta\psi}) - (g_{\mu\nu}dx^\mu dx^\nu)([\xi_a,\partial_{x^\eta}],\partial_{x^\psi}) - (g_{\mu\nu}dx^\mu dx^\nu)(\partial_{x^\eta},[\xi_a,\partial_{x^\psi}]) \\
&\quad = \xi_a(g_{\eta\psi}) \\
&\quad = \xi_a^\alpha \partial_{y^\alpha}(g_{\eta\psi}) = 0 \\
&\quad \iff g_{\eta\psi} = g_{\eta\psi}(x)
\end{aligned} \tag{40}$$

where it was used that $[\xi_a,\partial_{x^\eta}] = -(\partial_{x^\eta}\xi_a^\alpha)\partial_{y_\alpha}$ which implies $dx^\mu([\xi_a,\partial_{x^\eta}]) = 0$.

$$\begin{aligned}
&\mathcal{L}_{\xi_a}(g_{ab}(A^a_\mu dx^\mu + \omega^a)(A^b_\nu dx^\nu + \omega^b))(Y_c,Y_d) \\
&\quad = \xi_a(g_{ab}(A^a_\mu dx^\mu(Y_c) + \omega^a(Y_c))(A^b_\nu dx^\nu(Y_d) + \omega^b(Y_d)) \\
&\quad - (g_{ab}(A^a_\mu dx^\mu + \omega^a)(A^b_\nu dx^\nu + \omega^b)([\xi_a,Y_c],Y_d) \\
&\quad - (g_{ab}(A^a_\mu dx^\mu + \omega^a)(A^b_\nu dx^\nu + \omega^b)(Y_c,[\xi_a,Y_d]) \\
&\quad = \xi_a(g_{cd}) = \xi_a^\alpha \partial_{y^\alpha}(g_{cd}) = 0 \\
&\quad \iff g_{ab} = g_{ab}(x)
\end{aligned} \tag{41}$$

where it was used that $dx^\mu(Y_a) = 0$, $\omega^a(Y_b) = \delta^a_b$ and $[\xi_a, Y_b] = 0$. One then knows that $\mathcal{L}_{\xi_a} g_{cd} = 0$ and therefore

$$\begin{aligned}
&\mathcal{L}_{\xi_c}(g_{ab}(A^a_\mu dx^\mu + \omega^a)(A^b_\mu dx^\nu + \omega^b)) \\
&\quad = g_{ab}\mathcal{L}_{\xi_c}((A^a_\mu dx^\mu + \omega^a) \otimes (A^b_\mu dx^\nu + \omega^b)) = 0 \\
&\quad \implies \mathcal{L}_{\xi_c}(A^a_\mu dx^\mu + \omega^a) = 0
\end{aligned} \tag{42}$$

where the linearity of the Lie derivative and the Leibniz rule were used.

$$\begin{aligned}
&\mathcal{L}_{\xi_a}(A^b_\nu dx^\nu + \omega^b)(\partial_{x^\mu}) \\
&\quad = \xi_a(A^b_\mu) - (A^b_\nu dx^\nu + \omega^b)([\xi_a,\partial_{x^\mu}]) \\
&\quad = \xi_a(A^b_\mu) + \omega^b((\partial_{x^\mu}\xi_a^\alpha)\partial_{y^\alpha}) = 0 \\
&\quad \iff (\partial_{x^\mu}\xi_a^\alpha)\partial_{y^\alpha} = -\xi_a(A^b_\mu)\mathbf{Y}_b \ .
\end{aligned} \tag{43}$$

The y-dependence in A^b_μ is required to cancel out for the truncation. One therefore enforces $\partial_{x^\mu}\xi^\alpha_a = 0$ which implies

$$\xi_a = \xi^\alpha_a(y)\partial_{y^\alpha} \qquad \& \qquad A^a_\mu = A^a_\mu(x)\ . \tag{44}$$

One can then choose the vectors $\mathbf{Y}_a$ and the dual basis ω^a to be x-independent as well. This implies $\mathcal{L}_{\xi_a}\omega^b = 0$ and one can write down the truncated metric such that $\mathcal{L}_{\xi_a} g_R = 0$:

$$g_R = g_{\mu\nu}(x)dx^\mu dx^\nu + g_{ab}(x)(A^a_\mu(x)dx^\mu + \omega^a(y))(A^b_\nu(x)dx^\nu + \omega^b(y))\ . \tag{45}$$

There are no constraints that reduce the number of independent fields defining the theory. The original metric g depended on x and y and had $\frac{d(d+1)}{2} + \frac{n(n+1)}{2} + nd = \frac{(d+n)(d+n+1)}{2}$ independent components:

$$g = \begin{bmatrix} g_{\mu\nu}(x,y) & g_{ab}(x,y)A^a_\mu(x,y) \\ g_{ab}(x,y)A^b_\nu(x,y) & g_{ab}(x,y) \end{bmatrix} \tag{46}$$

The reduced metric g_R depends on x only, but still has $\frac{(d+n)(d+n+1)}{2}$ independent components:

$$g_R = \begin{bmatrix} g_{\mu\nu}(x) & g_{ab}(x)A^a_\mu(x) \\ g_{ab}(x)A^b_\nu(x) & g_{ab}(x) \end{bmatrix} \tag{47}$$

Thus, the number of degrees of freedom attached to every space-time point in configuration space is not changed through this type of truncation. Instead, the dimension of the space-time underlying the theory is reduced. The truncated metric is defined on a d-dimensional manifold transverse to the leaves of the foliation. Next, two propositions that will later be needed are proved.

Proposition 3: Maurer-Cartan-Equations

Let θ^a be a basis of left-invariant 1-forms. Then the following equation holds:

$$d\omega^a = \frac{1}{2}C^a_{bc}\omega^b \wedge \omega^c\ . \tag{48}$$

Proof. Let X_i be the dual basis to ω^a with $[X_i, X_j] = -C^k_{ij} X_k$

$$\begin{aligned} d\omega^a(X_i, X_j) &= -\omega^a([X_i, X_j]) \\ &= C^k_{ij}\omega^a(X_k) \\ &= C^a_{ij} = \frac{1}{2}(C^a_{ij} - C^a_{ji}) \\ \implies d\omega^a &= \frac{1}{2} C^a_{ij}\omega^i \wedge \omega^j . \end{aligned} \tag{49}$$

□

Since ω^a is x-independent, the exterior derivative on $d\omega^a$ only contains partial derivatives in y-direction and the Maurer-Cartan equation therefore holds for the dual vectors ω^a on the surfaces of foliation.

Proposition 4: Anholonomic derivative

$$\partial_\alpha(|\omega| Y^\alpha_a) = C^b_{ab}|\omega| \tag{50}$$

where $|\omega| = \det\ (\omega^a_\alpha)$

Proof. Express (48) in local coordinates:

$$d\omega^a(\partial_\alpha, \partial_\beta) = \partial_\alpha\omega^a_\beta - \partial_\beta\omega^a_\alpha = \frac{1}{2} C^a_{bc}(\omega^b_\alpha\omega^c_\beta - \omega^b_\beta\omega^c_\alpha) = C^a_{bc}\omega^b_\alpha\omega^c_\beta \tag{51}$$

Multiply both sides with Y^β_d, use that it is the inverse matrix of ω^a_α and take the trace:

$$\begin{aligned} (\partial_\beta\omega^a_\alpha)Y^\beta_a =& \partial_\beta(\delta^\alpha_\beta) - \omega^a_\alpha\partial_\beta Y^\beta_a \\ &= -\omega^a_\alpha(\partial_\beta Y^\beta_a)Y^\beta_a(\partial_\alpha\omega^a_\beta) + \omega^a_\alpha(\partial_\beta Y^\beta_a) \\ &= C^a_{ba}\omega^b_\alpha \end{aligned} \tag{52}$$

Jacobi's formula states that $\partial_\alpha|\omega| = \mathrm{tr}(\mathrm{adj}(\omega)\partial_\alpha\omega)$. Furthermore noting that $\mathrm{adj}(\omega) = |\omega|\omega^{-1} = |\omega|Y$, one can write

$$Y^\beta_a(\partial_\alpha\omega^a_\beta) = \frac{1}{|\omega|}\partial_\alpha(|\omega|)\ . \tag{53}$$

Saturating again with Y^α_d yields

$$\partial_\alpha(|\omega|)Y^\alpha_d + |\omega|(\partial_\alpha Y^\alpha_d) = \partial_\alpha(|\omega|Y^\alpha_d) = C^b_{db}|\omega|\ . \tag{54}$$

□

Let ϕ be a component of a field and let $\mathscr{L}$ be a Lagrangian density expressed in terms of the mixed basis:

$$\mathscr{L} = \mathscr{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \mathbf{Y}_b(\phi), \mathbf{Y}_a \mathbf{Y}_b(\phi)) \ . \tag{55}$$

The truncation is implemented by demanding the fields appearing in the Lagrangian to satisfy the Killing condition. Since the Lagrangian is a scalar density, it then also satisfies the Killing condition as a consequence. One needs to define a modified Lagrangian

$$\mathscr{L} =: |\omega| \tilde{\mathscr{L}} \tag{56}$$

since the dependences on y-coordinates in the Lagrangian are located in $|\omega|$ as shown in the following proposition. It is exactly these y-coordinates that need to be truncated.

Proposition 5: y-dependencies of Lagrangian

The Lagrangian $\mathscr{L}(x, y)$ is of the form $\mathscr{L}(x, y) = |\omega| f(x)$ for a scalar function $f(x)$.

Proof. $[\xi_a, Y_b] = 0$ implies $\xi_a^\alpha \partial_{y^\alpha} Y_b^\beta = Y_b^\alpha \partial_{y^\alpha} \xi_a^\beta \ \forall \beta$. Saturating with ω_β^b from the left and using (53) yields

$$\partial_{y^\beta} \xi_a^\beta = -Y_b^\beta \xi_a^\alpha \partial_{y^\alpha} \omega_\beta^b = -\frac{1}{|\omega|} \xi_a(|\omega|) = |\omega| \xi_a \left(\frac{1}{|\omega|} \right) \ . \tag{57}$$

Using $\mathcal{L}_{\xi_a} \mathscr{L} = \partial_\alpha (\mathscr{L} \xi_a^\alpha) = 0$ one obtains

$$\begin{aligned} \partial_\alpha (\mathscr{L} \xi_a^\alpha) &= \xi_a \mathscr{L} + \mathscr{L} |\omega| \xi_a \left(\frac{1}{|\omega|} \right) = |\omega| \xi_a \left(\frac{\mathscr{L}}{|\omega|} \right) = 0 \\ \implies \partial_\alpha \left(\frac{\mathscr{L}}{|\omega|} \right) &= 0 \implies \mathscr{L}(x, y) = |\omega| f(x) \ . \end{aligned} \tag{58}$$

□

At the level of the variational principle, the Lagrangian is truncated by setting the $\mathbf{Y}_a$-derivatives to zero and by demanding ϕ to satisfy the Killing condition:

$$\mathscr{L}_R(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi) := \tilde{\mathscr{L}}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \mathbf{Y}_b(\phi) = 0, \mathbf{Y}_a \mathbf{Y}_b(\phi) = 0) \ . \tag{59}$$

One then varies the action $\tilde{S} = \int \mathrm{d}^d x \ \mathrm{d}^n y \, \tilde{\mathscr{L}}$ and implements the truncation at the level of the e.o.m.. Analysing the difference between the two procedures, this section presents conditions for when the two procedures agree:

$$\begin{aligned}\delta\tilde{\mathscr{L}} =& \frac{\partial\tilde{\mathscr{L}}}{\partial\phi}\delta\phi + \frac{\partial\tilde{\mathscr{L}}}{\partial\phi_\mu}\delta\phi_\mu + \frac{1}{2}\frac{\partial\tilde{\mathscr{L}}}{\partial\phi_{\mu\nu}}\delta\phi_{\mu\nu} \\ &+ \frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\phi}\delta(\mathbf{Y}_a\phi) + \frac{1}{2}\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\mathbf{Y}_b\phi}\delta(\mathbf{Y}_a\mathbf{Y}_b\phi)\ .\end{aligned} \tag{60}$$

Integration by parts yields the Euler-Lagrange equations in the mixed basis. Note that the vector fields ω^a and $\mathbf{Y}_b$ are part of the basis and are therefore independent of the variation of the fields:

$$\int \mathrm{d}^d x\ \mathrm{d}^n y\,|\omega|\frac{\partial\tilde{\mathscr{L}}}{\partial\phi_\mu}\delta\phi_\mu = \text{div.} - \int \mathrm{d}^d x \mathrm{d}^n y\,|\omega|\partial_\mu\frac{\partial\tilde{\mathscr{L}}}{\partial\phi_\mu}\delta\phi \tag{61}$$

where it was used that ω does not depend on x and div. stands for the divergence terms that can be neglected because of Gauss' theorem. However, partial integration of the anholonomic basis terms produces extra terms compared to the partial integration of the holonomic basis terms:

$$\int \mathrm{d}^d x\ \mathrm{d}^n y\,\frac{1}{2}\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\phi}\delta(\mathbf{Y}_a^\alpha\partial_\alpha\phi) = \text{div.} + \int \mathrm{d}^d x\mathrm{d}^n y\,\frac{1}{2}\partial_\alpha\left(|\omega|Y_a^\alpha\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\phi}\right)\delta\phi\ . \tag{62}$$

Using proposition (4), one sees that $\partial_\alpha\left(|\omega|Y_a^\alpha\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\phi}\right) = |\omega|\left(C^b_{ab}\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\phi} + Y_a^\alpha\partial_\alpha\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\phi}\right)$. Hence, the Euler-Lagrange equations with field components expressed in the mixed basis are given as:

$$\begin{aligned}\delta\frac{\mathscr{L}}{|\omega|} =& \frac{\partial\tilde{\mathscr{L}}}{\partial\phi} - \partial_\mu\frac{\partial\tilde{\mathscr{L}}}{\partial\phi_\mu} \\ &+ \frac{1}{2}\partial_{\mu\nu}\frac{\partial\tilde{\mathscr{L}}}{\partial\phi_{\mu\nu}} - (\mathbf{Y_a} + C^c_{ca})\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\phi} \\ &+ \frac{1}{2}(\mathbf{Y_b} + C^c_{cb})(\mathbf{Y_a} + C^c_{ca})\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\mathbf{Y}_b\phi}\ .\end{aligned} \tag{63}$$

The Euler-Lagrange equations are truncated by setting the y-derivatives of ϕ to zero. The truncated e.o.m. are denoted with the subscript R and compared with the variation of the truncated Lagrangian (59):

$$\left(\frac{\delta\mathscr{L}}{\delta\phi}\right)_R = |\omega|\left(\frac{\delta\mathscr{L}_R}{\delta\phi} - C^c_{ca}\left(\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\phi}\right)_R + \frac{1}{2}C^c_{cb}C^c_{ca}\left(\frac{\partial\tilde{\mathscr{L}}}{\partial\mathbf{Y}_a\mathbf{Y}_b\phi}\right)_R\right)\ . \tag{64}$$

Remember that the truncation is called consistent if and only if the truncation at the level of the variation and at the level of the e.o.m. agree.

Theorem 3: Necessary and sufficient condition for consistent truncation

$$\begin{aligned}\left(\frac{\delta \mathscr{L}}{\delta \phi}\right)_R &= |\omega| \left(\frac{\delta \mathscr{L}_R}{\delta \phi}\right) \\ \iff C^c_{ac} \left(\frac{\partial \tilde{\mathscr{L}}}{\partial \mathbf{Y}_a \phi}\right)_R &= \frac{1}{2} C^c_{ac} C^d_{bd} \left(\frac{\partial \tilde{\mathscr{L}}}{\partial \mathbf{Y}_a \mathbf{Y}_b \phi}\right)_R\end{aligned} \tag{65}$$

Corollary 1: Unimodularity condition

$$C^c_{ac} = 0 \ \forall a \implies \left(\frac{\delta \mathscr{L}}{\delta \phi}\right)_R = |\omega| \left(\frac{\delta \mathscr{L}_R}{\delta \phi}\right) \tag{66}$$

The unimodularity condition is an invariant statement since the trace of the structure constants is invariant under a change of basis of the Lie algebra. It is equivalent to the statement that the adjoint representation of the isometry group is unimodular. Abelian, semi-simple and compact Lie groups are examples of such groups. In contrast with the abstract conditions in the Palais theorem, one can easily verify whether a given symmetry group allows for consistent truncation using theorem (3). Furthermore, it was shown that in the above setting, all components of the metric survive the truncation to the singlet sector under the action of the isometry group. One only imposes the condition that g must not depend on the coordinates y of the internal space.

3.2 Examples

This subsection considers two examples of truncation whose consistency can be decided on grounds of the unimodularity condition.

3.2.1 Kaluza Klein-reduction on $\mathbb{S}^1$

Consider the dimensional reduction of the (4+1)-dimensional Hilbert-Einstein-Lagrangian by assuming that the five-dimensional space-time has the form $\mathscr{M} \times \mathbb{S}^1$ where $\mathscr{M}$ is a 4-dimensional space-time manifold. This example is historically relevant because it was one of the first attempts to build a unified theory of gravity and electromagnetism. It also gave the process of *Kaluza-Klein dimensional reduction* its name. Kaluza (1921) postulated the existence of a fifth-dimension and considered five-dimensional Einstein gravity in 1919 shortly after the development of general relativity by Einstein[1]. In order to order to ensure unobservability of the fifth dimension, Kaluza introduced the assumption that there exists a coordinate system such that the metric tensor is independent of the extra coordinate z: $\partial_z g_{ab} = 0$. He realised that one could unify gravity with Maxwell theory in five dimensions at the cost of introducing an extra scalar field. Klein (1926) continued Kaluza's work by compactifying the five-dimensional theory on a circle and expanding the five-dimensional metric into a Fourier series

$$g_{ab} = \sum_{k=0}^{\infty} g_{\mu\nu}^{(k)}(x,z) e^{ikz/L} \tag{67}$$

with L being the diameter of the circle, x being the coordinates of the lower-dimensional space and z the coordinate on $\mathbb{S}^1$. Klein truncated the system by setting all the modes with $k \geq 1$ to zero and demanding $\partial_z g_{\mu\nu}^{(k)}(x,z) = 0$. The Fourier modes with $k \geq 1$ can be identified with massive modes where the parameter k labels their mass. One can see this by considering a massless scalar field ϕ in $d+1$ dimensions that satisfies $\partial^M \partial_M \phi = 0$ as toy model. Compactification, fourier expansion and enforcement of the Killing condition together yield

$$\phi(x,z) = \sum_{k=0}^{\infty} \phi_k(x,z) e^{ikz/L} \ . \tag{68}$$

The four-dimensional fields $\phi_{(k)}$ satisfy

$$\partial^M \partial_M \phi_k - \frac{k^2}{L^2} \phi_k = 0 \ . \tag{69}$$

This is the wave equation for a scalar field of mass $\frac{|k|}{L}$. The identification of the conserved charge of the first massive mode stemming from invariance of the system under $z \mapsto z' = Cz + \epsilon(x)$ with $C =$ const. with the elementary electric charge e allowed Klein to determine the radius of the circle. He had at once explained the quantisation of the electric charge and the size of the compact dimension of the order of the Plank length before the development of QM. However, he obtained a wrong result for a mass of such a mode, that was at the order of the Planck mass.

At that time, Klein did not bother about the consistency of such an ansatz. The philosophy he adhered to was that the mass of the massive modes was so high that one would not be able to observe these modes at energies currently accessible.

[1] The first reference to dimensional reduction appears in the work of Nordström (1914). Starting from Maxwell theory in a five-dimensional flat spacetime, Nordström constructed a vector-scalar theory in four dimensions unifying electromagnetism and a scalar theory of gravitation.

Yet, if one ignores the issue of consistency, one basically denies the physical reality of the higher-dimensional space. This has become a philosophically unattractive thing to do. In addition, one wants to maintain consistency with the higher dimensional field equations at every stage on the derivation of the effective lower-dimensional field theory in order to have mathematical control over the solution.

Fortunately, the truncation scheme that Klein employed is also consistent in the sense presented in this thesis. He assumed that the five-dimensional spacetime would admit $U(1)$ as isometry group. Truncation to the massless sector ($k = 0$) and demanding $g_{\mu\nu} = g_{\mu\nu}(x)$ is then equal to implementing the Killing condition $\mathcal{L}_{dz} g = 0$ on the metric where dz is the Killing vector of $\mathbb{S}^1$. Since $U(1)$ is compact and therefore unimodular, the unimodularity condition applies and the truncation is consistent.[2]

3.2.2 Homogeneous Bianchi cosmology

This section studies the inconsistency arising in the dimensional reduction of the Hilbert-Einstein Lagrangian in four dimensions under the action of a three-dimensional Lie group that is not unimodular. Hawking (1969) first realised that one could not derive the correct field equation from the reduced Hilbert-Einstein Lagrangian in this setting.

Definition 8: Spatially homogeneous spacetime $(\mathscr{M}, g_{ab})$

A spatially homogeneous spacetime is a pseudo-Riemannian manifold $\mathscr{M}$ with a family of spacelike hypersurfaces Σ_t such that $\forall p, q \in \Sigma_t$, there exists an element $g : \mathscr{M} \to \mathscr{M}$ with $g \in g$ such that $g(p) = q$. G then acts *transitively* on each Σ_t. If the element g is unique $\forall p, q \in \Sigma_t$, the action of G is called *simply transitive*.

Consider a spatially homogeneous cosmological model with a three-parameter group of isometries that foliate the space-time into three-dimensional, invariant, spacelike hypersurfaces. As argued in Ryan and Shepley (1975), there is no loss of generality in assuming that the action is simply transitive. The space-like hypersurfaces are labeled by a parameter t such that

$$g^{AB} t_{;A} t_{;B} = -1 \ . \tag{70}$$

[2] Note that Klein himself kept the additional scalar field, also called the dilaton field, appearing in the lower-dimensional theory. However, many publications following him over the years, incorrectly set it constant. This is in conflict with the five-dimensional Einstein field equations and an example of inconsistent truncation.

The vectors $t_{;A}$ are timelike and normal to the spacelike hypersurfaces. As in the derivation of the unimodularity condition, each surface of homogeneity supports a Lie algebra $\mathfrak{g}$ spanned by the Killing vector fields ξ_i such that

$$[\xi_i, \xi_j] = -C^k_{ij}\xi_k \ . \tag{71}$$

One can choose an arbitrary basis of the tangent space at the identity element of each foliation leaf, extend it to a basis vector field Y_i on each foliation surface and define a dual left invariant basis σ^i such that $\sigma^i(Y_j) = \delta^i_j$. The vectors Y_i are of course independent of t and only depend on the coordinates of the spatial hypersurfaces. They satisfy the commutation relation

$$[Y_i, Y_j] = C^k_{ij} Y_k \ . \tag{72}$$

The metric can now be expressed in the basis of 1-forms $\{-dt, \sigma^i\}$:

$$ds^2 = -dt^2 + g_{ij}(t)\sigma^i\sigma^j \tag{73}$$

where g_{ij} is a 3×3-matrix which depends only on t after implementation of the Killing condition

$$\mathcal{L}_{\xi_i} g = 0 \implies g_{ab} = g_{ab}(t) \ . \tag{74}$$

Lower-case Latin indices may be lowered and raised using $\gamma_{ab} := g_{ab} = g_{AB}\sigma^A_a\sigma^B_b = Y^i_a Y_{bi}$ and its inverse $\gamma^{ab} = g^{ab} = g^{AB}\sigma^a_A\sigma^b_B = Y^a_i Y^{ib}$. Denote the vector field basis dual to $\{-dt, \sigma^i\}$ by X_A. The vector fields X_A satisfy:

$$\begin{aligned} [X_0, X_i] &= 0 \\ [X_i, X_j] &= -C^k_{ij} X_k \\ X_0 \cdot X_0 &= -1 \\ X_0 \cdot X_i &= 0 \\ X_i \cdot X_j &= g_{ij}(t) \ . \end{aligned} \tag{75}$$

X_i is called the *synchronous basis* and is unique as long as the homogeneous hypersurfaces remain spacelike (Ryan & Shepley, 1975). Let θ^μ be an orthonormal basis defined via $\theta^0 = dt$; $\theta^i = B_{is}(t)\sigma^s$ where the matrix $B = (B_{ij})$ is the symmetric square root of $G = (g_{ij})$ with $b_{is}b_{sj} = g_{ij}$. The metric $ds^2 = \eta_{\mu\nu}\theta^\mu\theta^\nu$

looks Minkowskian in this basis. The exterior derivatives of θ^μ are given as:

$$
\begin{aligned}
d\theta^0 &= 0 \\
d\theta^i &= (\dot{B_{is}})\theta^0 \wedge \sigma^s + B_{is} d\sigma^s \\
&= (\dot{B_{is}})B^{sj}\theta^0 \wedge \theta^j + \frac{1}{2} B_{ij} B^{sk} B^{tm} C^j_{st} \theta^k \wedge \theta^m \\
&= k_{ij}\theta^0 \wedge \theta^j + \frac{1}{2} d^i_{km} \theta^k \wedge \theta^m
\end{aligned}
\tag{76}
$$

where a dot denotes differentiation with respect to t, $B^{ij} := (B^{-1})_{ij}$ and

$$
\begin{aligned}
k_{ij} &:= (\dot{B_{is}})B^{sj} \\
d^i_{km} &:= B_{ij} B^{sk} B^{tm} C^j_{st}
\end{aligned}
\tag{77}
$$

were defined. Further, it was used that the dual vector fields σ^a satisfy the Maurer-Cartan equation (48). One notes that d^i_{km} have the same symmetry as C^i_{km} and also satisfy the *Jacobi equation*. The connection 1-forms ω^μ are computed using *Cartan's first structure equation*:

$$
\Theta^\mu = d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu \ , \tag{78}
$$

Using vanishing torsion $\Theta^\mu = 0$, one obtains:

$$
\omega^0_j \wedge \theta^j = 0 k_{is}\theta^0 \wedge \theta^s + \frac{1}{2} d^i_{st}\theta^s \wedge \theta^t + \omega^i_j \wedge \theta^j = 0 \tag{79}
$$

The components of the connection 1-forms in the orthonormal basis are called *Ricci rotation coefficients* $\omega^\mu_{\nu\eta} := \omega^\mu_\nu(\theta_\eta)$. Using the antisymmetry of $\omega_{\mu\nu}$, one obtains the following equations for the Ricci rotation coefficients:

$$
\begin{aligned}
k_{ij} &= \omega^i_{0j} - \omega^i_{j0} \\
d^i_{st} &= \omega^i_{st} - \omega^i_{ts} \\
0 &= \omega^0_{\mu\nu} - \omega^0_{\nu\mu}
\end{aligned}
\tag{80}
$$

and therefore:

$$
\begin{aligned}
\omega^0_{i0} &= 0 \\
\omega^0_{ij} = \omega^0_{ji} &= k_{(ij)} \\
\omega^i_{j0} &= -k_{[ij]} \\
\omega^k_{ij} &= \frac{1}{2}(d^k_{ij} - d^j_{ki} - d^i_{kj}) \ .
\end{aligned}
\tag{81}
$$

The connection forms ω are completely determined by these conditions on the

Ricci rotation coefficients:

$$\begin{aligned} \omega^0_i &= k_{(ij)}\theta^j \\ \omega^i_j &= -k_{[ij]}\theta^0 + \frac{1}{2}(d^i_{js} - d^j_{is} - d^s_{ij})\theta^s \ . \end{aligned} \tag{82}$$

Using that $d^i_{km} = B_{ij}B^{sk}B^{tm}C^j_{st}$ in the θ-basis, one sees that in the σ-basis of the spatial hypersurfaces:

$$\omega^k_{ij} = \frac{1}{2}(C^k_{ij} - C^j_{ki} - C^i_{kj}) \ . \tag{83}$$

This is an important and completely general result for the connection coefficients of homogeneous spaces. It is now used to illustrate the inconsistency arising in the variation of the reduced Einstein-Hilbert-Lagrangian with respect to the metric components on the spatial hypersurfaces g_{ij}. Note that the form of the metric does not allow g_{00} and g_{0i} to be varied. One can only derive 6 out of 10 field equations via variation. If one did vary the remaining metric components, the variation would not respect the symmetry assumptions which force g_{AB} into the product form $ds^2 = -dt^2 + g_{ij}\sigma^i\sigma^j$ with g_{ij} being the components of the spatial metric. Furthermore, $g_{ij} = g_{ij}(t)$ implies that one can choose the region in which the metric is varied to be bounded in time, but it may be unbounded in space. The crux of the issue lies in requiring the variations to satisfy some symmetry conditions. One will obtain the correct e.o.m. if and only if the divergence obtained in the variation of $\mathcal{R}$ vanishes identically. Following Sneddon (1976), it is shown that this is only the case if the trace of the structure constants vanishes. Consider the Einstein Hilbert action in four dimensions:

$$S = \int_{V'} \sqrt{-g}R_{AB}g^{AB}\mathrm{d}^4x \ . \tag{84}$$

R_{AB} and g^{AB} denote the four-dimensional Ricci tensor and metric. V' is chosen to be a compact region bounded by a closed surface. Otherwise, the integral will diverge because the spatial hypersurfaces are not unimodular and therefore not compact. Variations are made by considering that the one-parameter family of space-times differ only within V'. R_{AB} and g^{AB} are functions of t only because of the symmetry condition. Therefore, the action integral is only t dependent and can be expressed as

$$S = \int \mathcal{L}(t)\mathrm{d}t \left(\int L(x^i)\,\mathrm{d}^3x\right) \ . \tag{85}$$

Looking at the sum in (85), one sees that the term $\sqrt{-g}R_{ab}g^{ab} =: \sqrt{-g}R^{(3)}$ appears in the sum where $R^{(3)}$ is the Ricci scalar on the three-dimensional hypersurfaces. The inconsistency arises in the variation of $R^{(3)}$ with respect to g^{ab}:

$$\begin{aligned}\delta \int R_{ab} g^{ab} \sqrt{-g}\, \mathrm{d}t = &- \int \sqrt{-g} R_{ab} \delta g_{ab}\, \mathrm{d}t \\ &+ \frac{1}{2} \int \sqrt{-g} R_{cd} g^{cd} g^{ab} \delta g_{ab}\, \mathrm{d}t \\ &+ \int g^{ab} \sqrt{-g} \delta R_{ab}\, \mathrm{d}t \end{aligned} \tag{86}$$

where it was used that the variation of the metric components g_{00} and g_{0i} in the metric determinant with respect to g_{ab} vanishes and exploited $g^{ab}\delta g_{ab} = -g_{ab}\delta g^{ab}$ as well as *Cramer's rule* to compute $\delta\sqrt{-g} = -\frac{\sqrt{-g}}{2} g_{ab}\delta g^{ab}$. The connection coefficients on the spatial hypersurfaces in an arbitrary basis are given as

$$\Gamma^{d}_{ba} = \frac{1}{2} g^{dc} (\omega_{cab} + \omega_{bca} - \omega_{abc} + g_{ca;b} + g_{bc;a} - g_{ab;c}) \tag{87}$$

where ω_{ijk} are the Ricci rotation coefficients. The results for coordinate bases (where $\omega_{ijk} = 0$) and for orthonormal bases (where $g_{ab;c} = \eta_{ab;c} = 0$) follow from (87) as special cases. The components of the Ricci tensor can now be expressed as usual using the general connection coefficients:

$$R_{ab} = \Gamma^{c}_{ab;c} - \Gamma^{c}_{ac;b} + \Gamma^{c}_{dc}\Gamma^{d}_{ab} - \Gamma^{d}_{ac}\Gamma^{c}_{bd}\ . \tag{88}$$

The introduction of normal coordinates allows to ignore terms quadratic in the connection coefficients:

$$R_{ab} = \Gamma^{c}_{ab;c} - \Gamma^{c}_{ac;b}\ . \tag{89}$$

There has been some confusion in the literature over whether this derivation holds in a non-coordinate frame (Ryan, 1974). The reason of this confusion is that the connection coefficients are not frame-independent quantities. However, one can show that the variation of the connection coefficients transforms tensorially which is why the standard derivation of the field equations is valid in arbitrary bases. Upon variation of the Ricci tensor, one obtains the *Palatini identity*:

$$\begin{aligned} \delta R_{ab} &= \nabla_c(\delta\Gamma^{c}_{ab}) - \nabla_b(\delta\Gamma^{c}_{bc}) \\ g^{ab}\delta R_{ab} &=: \partial_a W^a = \nabla_a W^a \\ W^a &= g^{bd}(\delta\Gamma^{a}_{bd}) - g^{ab}(\delta\Gamma^{c}_{bc}) \\ &= (g^{ad}g^{cb} - g^{ab}g^{dc})(\delta g_{cd})_{;b} \end{aligned} \tag{90}$$

If δg_{ab} and $(\delta g_{ab})_{;c}$ vanish at the integration boundaries, the variation of the divergence vanishes by Gauss' theorem and one obtains the usual vacuum field equations. However, the ansatz (73) entails that g_{ab} and δg_{ab} are functions of the time t only. Therefore, δg_{ab} and $(\delta g_{ab})_{;c}$ cannot be made to vanish at the integration boundaries without δg_{ab} vanishing everywhere and the variation becoming trivial. In this case, the divergence of W^a itself must vanish in order for the integral to vanish. Using the antisymmetry of the structure constants, it follows that:

$$\begin{aligned} \Gamma^c_{ab} &= \frac{1}{2} g^{cg}(C_{gab} + C_{bga} - C_{abg}) \\ \nabla_i(\sigma^j)^i = -\Gamma^j_{ab}\sigma^{ai}\sigma^b_i &= -\frac{1}{2} g^{ab}(g^{jc}(C_{cab} + C_{bca} - C_{abc})) \\ &= -g^{jc}C^a_{ca} \\ \nabla_a W^a &= \nabla_a(W_i\sigma^i)^a = W_i\nabla_a(\sigma^i)^a = -W_i g^{ic} C^b_{cb} \\ &= W^a C^b_{ba} \ . \end{aligned} \tag{91}$$

One concludes

$$\int \nabla_a W^a \sqrt{g}\, \mathrm{d}t = \int W^a C^b_{ba} \sqrt{g}\, \mathrm{d}t \neq 0 \ . \tag{92}$$

if the unimodularity condition derived earlier is not met. MacCallum and Taub (1972) showed that if $\int \sqrt{g}\, \nabla_a W^a\, \mathrm{d}t = 0$ is to hold for arbitrary δg_{ab} this implies $C^b_{ba} = 0$. Assembling all the pieces, one obtains:

$$\delta \int \sqrt{g} R^{(3)}\, \mathrm{d}t = \int G_{ab}\delta g^{ab}\eta + \int W^a C^b_{ba}\sqrt{g}\, \mathrm{d}t \neq 0 \tag{93}$$

for the variation of the action on the surfaces of homogeneity in a non-unimodular reduction. One cannot derive the correct field equations from the reduced action and the truncation is therefore inconsistent.

4

Coset space reductions

Chapter 3 considered dimensional reductions where the isometry group G is generated by a set of dim G linearly-independent Killing vectors. If one consider a dimensional reduction through a spacetime of the form $\mathscr{M} \times \mathbb{S}^n$ admitting $SO(n+1)$ as isometry group for instance, the condition of linearly independent Killing vectors is not met in general. Yet, as argued earlier, *Palais' theorem* guarantees consistent truncation to the set of symmetric points on spheres. This chapter adapts the proof of the *unimodularity condition* given in chapter 3 to hold for the case of dimensional reductions on *coset spaces*. This includes the case of dimensional reductions on spheres because $\mathbb{S}^n = SO(n+1)/SO(n)$. The main result of this chapter is that in order for the metric g to be a symmetric element, it is not sufficient to require the variable dependencies of the metric to be of the form $g_{AB} = g_{AB}(x)$ where x are the coordinates surviving the truncation. This is because a left coset G/H admits a global left action of its isometry group G while it does usually not admit a global right action of G. Hence, one will be hard-pressed 1-forms on G/H that are invariant under the action of G with which to express the metric. One way to deal with this difficulty is to impose the additional constraints

$$C^c_{ia} g_{cb} + C^c_{ib} g_{ac} = 0 \qquad A^b_\nu = 0 \tag{94}$$

on the metric. They imply $\mathcal{L}_{\xi_a}\mathbf{g} = 0 \iff \mathbf{g} \in \Sigma$ and the truncation is consistent if the unimodularity condition is met. The following section on the geometry of coset spaces is based on Castellani (1999), Kapetanakis (1992) and Salam and Strathdee (1982).

4.1 Left invariant 1-forms on the coset

Remember that in the case of independent Killing vectors, one could simply choose a basis of the Lie algebra on each foliation surface, left-translate it to obtain left-invariant basis vector fields and take their dual vector fields as dual basis on the foliation leaves. The goal of this section is to identify a suitable dual basis of the coset space G/H that comes as close to the group manifold reduction case as possible.

Let G be a Lie group and let $H \subseteq G$ be a Lie subgroup with H acting on G by *right* translations. The orbits of $g \in G$ are the *left* cosets of H in G with respect to g: $gH = \{gh|h \in H\}$. The space of left cosets is denoted by $G/H = \{gH|\ g \in G\}$. The left action of G on G/H given via $\mu_L : G \times G/H \to G/H;\ (g, g'H) \mapsto (gg')H$ generates isometries on G/H. It is important to note that one cannot define a canonical right action of G on G/H.

The following index conventions are used in this section: lower-case Latin alphabets denote tangent space indices ($i, j, k = 1$, ... dim $\mathfrak{h}$); ($a, b, c =$ dim $\mathfrak{h}$ +1, ..., dim $\mathfrak{g}$); ($A, B, C = 1$, ..., dim $\mathfrak{g}$); and lower-case Greek alphabets denote coset space coordinates ($\alpha, \beta, \gamma =$ dim $\mathfrak{h}$ +1, ..., dim $\mathfrak{g}$) Note that dim $G/H =$ dim G - dim $H =$ dim $\mathfrak{g}$ - dim $\mathfrak{h}$.

The Lie algebra $\mathfrak{g}$ of G can be split as $\mathfrak{g} = \mathfrak{h} \otimes \mathfrak{K}$ where $\mathfrak{h}$ is the Lie algebra of H with generators Q_i and $\mathfrak{K}$ contains the remaining generators Q_a, called coset generators. The structure constants of G are defined by

$$\begin{aligned} [Q_i, Q_j] &= C^k_{ij} Q_k \qquad Q_i \in \mathfrak{h} \\ [Q_i, Q_a] &= C^j_{ia} Q_j + C^b_{ia} Q_b \qquad Q_a \in \mathfrak{K} \\ [Q_a, Q_b] &= C^j_{ab} Q_j + C^c_{ab} Q_c \ . \end{aligned} \tag{95}$$

For compact or semisimple H, there exists a set of K_a such that the structure constants C^j_{ia} vanish. In this case the Lie algebra $\mathfrak{g}$ of G can be decomposed into a direct sum $\mathfrak{g} = \mathfrak{h} + \mathfrak{K}$ such that $[\mathfrak{h}, \mathfrak{h}] \in \mathfrak{h}$ $[\mathfrak{h}, \mathfrak{K}] \in \mathfrak{K}$. The coset space G/H is called *reductive* in this case. Reductivity of $\mathfrak{g}$ implies that its structure constants C^a_{ib} can be made antisymmetric in a and b by a redefinition of K_a. In the following analysis, G/H will be assumed to be reductive and C^a_{ib} to be antisymmetric in a and b. The coordinates of G/H are labeled by y^α. A representative L_y is chosen for each coset $L_y := \exp[y^\alpha \delta^a_\alpha Q_a]$ via exponentiation of the Lie algebra with the coset coordinates y^α. Later, it is shown later that the following constructions are independent of the choice of representative. Multiplication by $g \in G$ will carry L_y into another coset with representative $L_{y'}$. Then L_y and $L_{y'}$ are related by an extra transformation $h \in H$:

$$gL_y = L_{y'}h \qquad \text{for } h \in H \ . \tag{96}$$

(96) shows that $g \in G$ and y^α determine $h \in H$ and y'^α. Defining $h(z) := \exp[z^i Q_i]$ where z^i are labels along each coset, one can express $g \in \mathbf{G}$ as

$$\begin{aligned} g &= \exp[y^\alpha \delta^a_\alpha Q_a] \exp[z^i Q_i] \\ &= L_y h(z) \ . \end{aligned} \tag{97}$$

One extracts the dual basis by considering a suitable 1-form that takes values in $\mathfrak{g}$.

Definition 9: $\mathfrak{g}$-valued p-forms on a manifold M

A general $\mathfrak{g}$-valued p-form on $U \subset M$ is given as

$$\phi = E_R \otimes \phi^R \tag{98}$$

where ϕ^R is a p-form on U and E_R is a basis of $\mathfrak{g}$. Let X be a tuple of p tangent vectors. Then $\phi(X) = E_R \phi^R(X) \in \mathfrak{g}$. Let $\psi = E_S \otimes \psi^S$. One defines

$$d\phi := E_R \otimes d\phi^R \qquad [\phi, \psi] := [E_R, E_S] \otimes \phi^R \wedge \psi^R \ . \tag{99}$$

Definition 10: Maurer-Cartan form

The Maurer-Cartan form ω is a distinguished Lie-algebra-valued 1-form on a Lie group G. Let $\{E_R\}$ be a basis of $\mathfrak{g}$ and let $\{X_R\}$ be the left-invariant fields on G obtained by left translating the vectors E_R. Let $\{\sigma^R\}$ be left invariant 1-forms on G forming at each $g \in G$ a basis dual to $\{X_R\}$. Then

$$\omega := E_R \otimes \sigma^R \tag{100}$$

defined by

$$\omega(Y_g) = E_R \sigma^R(Y_g) = E_R Y^R \tag{101}$$

takes a vector $Y = X_R Y^R$ at $g \in G$ and left translates it back to the identity.

The following very common notation will be used

$$\omega = g^{-1} dg := (L_{g^{-1}})_* \circ dg \tag{102}$$

where L_g is left translation and dg is the vector valued 1-form at $g \in G$ that takes each vector Y at g into itself $dg := \frac{\partial}{\partial x^A} \otimes \mathrm{d}x^A$. In the next step, consider the Lie-algebra valued 1-form

$$e(y) := L_y^{-1} dL_y \tag{103}$$

on G/H. It is the *equivalent of the Maurer-Cartan form* on coset spaces. Since $e(y)$ takes values in $\mathfrak{g}$, it can be expanded in terms of the generators of G:

$$e(y) = e^a(y) Q_a + e^i(y) Q_i \ . \tag{104}$$

Yet, unlike the Maurer-Cartan form on a group manifold, $e(y)$ it is not left-invariant under the action of G:

$$\begin{aligned} e(y) \mapsto e(y') = L_{y'}^{-1} d(L_{y'}) &= h L_y^{-1} g^{-1} d(g L_y h^{-1}) \\ &= h e(y) h^{-1} + h dh^{-1} + h L_y^{-1} g^{-1} dg L_y h^{-1} \ . \end{aligned} \tag{105}$$

The matrices D^a_b of the adjoint representation of G are introduced

$$g^{-1} X_a g = D^b_a(g) X_b \tag{106}$$

. The action of G on $e(y)$ can be expressed as

$$e^a(y') = D^a_b(h^{-1}) e^b(y) + (h dh^{-1})^a + D^a_b(L_y h^{-1})(g^{-1} dg)^b \ . \tag{107}$$

The 1-forms $e^a(y) = e^a_\alpha(y)\mathrm{d}y^\alpha$ form a dual vielbein basis on G/H. They can be used to express the metric tensor. However, as can be seen from to projection of the transformation rule (107) to the coset generators Q_a, they are not left-invariant as in the independent Killing vector case, but transform as

$$e^a(y) \mapsto e^a(y') = D^a_b(h^{-1}) e^b(y) \tag{108}$$

Studying the situation by considering an infinitesimal G-transformation g in the neighborhood of the identity yields

$$g = 1 + \epsilon^A Q_A = 1 + \epsilon^i Q_i + \epsilon^a Q_a \tag{109}$$

for infinitesimal parameters ϵ^A as well as

$$h = 1 - \epsilon^A W^i_A(y) Q_i \tag{110}$$

where $W^i_A(y)$ is called the *H-compensator*. The coset coordinates y^α transform as

$$\delta y^\alpha = y'^\alpha - y^\alpha = \epsilon^A \xi_A^\alpha(y) \tag{111}$$

where $\xi_A^\alpha(y)$ the components of the Killing vector fields on the coset. These Killing vectors $\xi_A(y)$ associated with the generators Q_A are given by

$$\xi_A(y) := \xi_A^\alpha(y) \frac{\partial}{\partial y^\alpha} \ . \tag{112}$$

(96) implies

$$\xi_a^\alpha \approx \delta_a^\alpha - \frac{1}{2} y^\beta \delta_\beta^b C_{ab}^c \delta_c^\alpha \qquad \xi_i^\alpha \approx -y^\beta \delta_\beta^a \delta_c^\alpha C_{ia}^c \tag{113}$$

to linear order in y^α. This expression will prove very useful to compute the action of the Lie derivative with respect to the Killing vectors ξ_A. They obey the algebra

$$[\xi_A, \xi_B] = C_{AB}^C \xi_C \ . \tag{114}$$

Writing (107) for infinitesimal g, one obtains

$$X_A L_y = \xi_A(y) L_y - L_y Q_i W_A^i(y) \ . \tag{115}$$

Multiplication by $L^{-1}(y)$ from the left and projection on the K-generators to find an algebraic expression for the Killing vectors yields

$$\xi_A^\alpha(y) = D_A^a(L_y) e_a^\alpha(y) \ . \tag{116}$$

One can give an explicit expression for the left action of G on the 1-forms basis e^a noting that the C_{ib}^a are the generators of the adjoint representation of H and that $C_{ij}^a = 0$:

$$e^a(y') - e^a(y) = -\epsilon^A W_A^j(y) C_{jb}^a e^b(y) \ . \tag{117}$$

The metric of the coset space G/H can be expressed in terms of the basis of 1-forms as

$$\begin{aligned} g &= g_{\alpha\beta}(y) \mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta \\ &= \delta_{ab} e^a \otimes e^b \\ &= \delta_{ab} e_\alpha^a(y) e_\beta^b(y) \mathrm{d}y^\alpha \otimes \mathrm{d}y^\beta \end{aligned} \tag{118}$$

where $e_c^a(y)$ are the components of the 1-forms $e^a(y)$. (108) and (118) reveal the general form of a G- invariant metric on G/H:

$$g_{ab} = g_{cd} D_a^c(h^{-1}) D_b^d(h^{-1}) \ . \tag{119}$$

Expressed infinitesimally this is equivalent to:

$$C^b_{ia} g_{db} + C^b_{id} g_{ab} = 0 \tag{120}$$

This is the *invariance condition* of the general metric of the coset space under G-transformations.

4.2 Metric reduction

Let x be coordinates parameterising a d-dimensional manifold $\mathscr{M}$ and let G/H be an n-dimensional coset space. The index conventions mostly agree with the conventions in Chapter 3 with the difference that lower-case Greek alphabets α, β, γ are still used to distinguish coset coordinate indices when necessary. Lower-case Greek alphabets $\mu, \nu, \eta = 1, ..., d$ denote the lower-dimensional manifold, lower-case Latin alphabets $a, b, c, ... = d+1, ..., d+n$ denote the coset space, upper-case Latin alphabets $A, B, C, ... = 1, ..., d+n$ denote the product manifold $\mathscr{M} \times G/H$. The coordinates of the product manifold are denoted as $x^M = (x^\mu, y^a)$. The action of G on $\mathscr{M} \times G/H$ is defined via

$$g(x)^M = (x^\mu, g(y)^a), \quad g \in G \ . \tag{121}$$

The metric of $\mathscr{M} \times G/H$ is now expressed in exactly the same way as in the case of independent Killing vectors:

$$\begin{aligned} g =& g_{\mu\nu}(x,y)\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &+ g_{ab}(x,y)(A^a_\mu(x,y)\mathrm{d}x^\mu + e^a(y))(A^b_\nu(x,y)\mathrm{d}x^\nu + e^b(y)) \end{aligned} \tag{122}$$

If the metric satisfies the Killing condition $\mathcal{L}_{\xi_A} g = 0$, the proof for the unimodularity condition in the last chapter applies. Note however that this section requires the further restriction that the algebra $\mathfrak{g}$ be reductive. The components of the metric tensor g_{AB} can be grouped into:

- the metric $g_{\mu\nu}$ on $\mathscr{M}$ which transforms as scalar under G-transformations
- $g_{\mu a}$ and $g_{a\mu}$ which transform as vectors under G-transformations
- the metric g_{ab} on G/H which transforms as second-rank tensor under G-transformations

First, consider a G-invariant scalar field ϕ. It obeys

$$\phi(x, g(y)) = \phi(x, y) \ \forall g \in G \tag{123}$$

from which it follows that $\phi = \phi(x)$ and therefore $g_{\mu\nu} = g_{\mu\nu}(x)$ just as in the group manifold reduction case.

Secondly, consider the action of G on the metric tensor g_{ab} on G/H. The action of G will be expressed using the Lie derivative with respect to the Killing vectors generating G. Since the action of G on G/H is transitive, the value of a symmetric field at any point on G/H is determined by its value at the origin and a G-transformation. Therefore, one can conveniently calculate the Lie derivative at $y = 0$. Using the coordinate expression of the Killing vectors (113), it follows that

$$\begin{aligned}\mathcal{L}_{\xi_i}(g_{cd}) &= g_{\gamma d}\partial_c\xi_i^\gamma + g_{c\gamma}\partial_d\xi_i^\gamma \\ &= g_{\gamma d}(-\frac{1}{2}\delta_c^\beta\delta_\beta^b\delta_e^\gamma C_{ib}^e) + g_{c\gamma}(-\frac{1}{2}\delta_d^\beta\delta_\beta^b\delta_e^\gamma C_{ib}^e) \stackrel{!}{=} 0 \\ &\implies C_{ic}^e g_{ed} + C_{id}^e g_{ec} = 0\ .\end{aligned} \tag{124}$$

This is exactly the infinitesimal invariance condition (120). Taking the derivative with respect to ξ_a yields

$$\begin{aligned}\mathcal{L}_{\xi_a}(g_{cd}) &= \xi_a^\gamma(\partial_\gamma g_{cd}) + g_{\gamma d}\partial_c\xi_a^\gamma + g_{c\gamma}\partial_d\xi_a^\gamma \\ &= \partial_a g_{cd} + g_{\gamma d}\partial_c\xi_a^\gamma + g_{c\gamma}\partial_d\xi_a^\gamma \\ &= \partial_a g_{cd} + g_{\gamma d}(-\frac{1}{2}\delta_c^\beta\delta_\beta^b\delta_e^\gamma C_{ab}^e) + g_{c\gamma}(-\frac{1}{2}\delta_d^\beta\delta_\beta^b\delta_e^\gamma C_{ab}^e) \stackrel{!}{=} 0 \\ &\implies \partial_a g_{cd} + C_{ac}^e g_{ed} + C_{ad}^e g_{ec} = 0\end{aligned} \tag{125}$$

Thirdly, consider the action of the Lie-derivative on a G-invariant vector A_α:

$$\begin{aligned}\mathcal{L}_{\xi_a}A_\alpha &= \xi_a^\beta\partial_\beta A_\alpha + \partial_\alpha\xi_a^\beta A_\beta \\ &\stackrel{y=0}{=} \delta_a^\beta\partial_\beta A_\alpha - \frac{1}{2}\delta_\alpha^\beta\delta_\beta^b C_{ab}^c\delta_c^\alpha A_\beta \\ &= \partial_a A_\alpha - \frac{1}{2}C_{a\alpha}^c A_c = 0 \\ \mathcal{L}_{\xi_i}A_\alpha &= -\delta_\alpha^\gamma\delta_\gamma^a\delta_c^\beta C_{ia}^c A_\beta \\ &\stackrel{y=0}{=} -C_{i\alpha}^c A_c = 0\end{aligned} \tag{126}$$

This allows the conclusion that $\partial_a A_b = \frac{1}{2}C_{ab}^c A_c$ and $C_{ia}^b A_b = 0$. Since C_{ib}^c can be regarded as H-generator within G, A_b will be zero unless it is a singlet under the action of H. These conditions on the vectors $g_{ab}(x,y)A_\mu^a(x,y)$ for fixed μ can in general be satisfied by enforcing

$$A_\nu^b = 0\ . \tag{127}$$

Thus, implementation of the Killing condition forces the metric on $\mathscr{M} \times M_G$

$$g = \begin{bmatrix} g_{\mu\nu}(x,y) & g_{ab}(x,y)A^a_\mu(x,y) \\ g_{ab}(x,y)A^b_\nu(x,y) & g_{ab}(x,y) \end{bmatrix} \tag{128}$$

into the form of a product metric

$$g_R = \begin{bmatrix} g_{\mu\nu}(x) & 0 \\ 0 & ((g_{\alpha\beta})_R)(x) \end{bmatrix} \tag{129}$$

where $(g_{\alpha\beta})_R$ is the left-invariant coset metric. After implementing the Killing condition, the left action of G on the dual basis $e^a(y)$ is equivalent to an $SO(n)$ rotation of $e^a(y)$ because C^a_{ib} was chosen to be antisymmetric in a and b. The natural coset metric $(g_{\alpha\beta})_R$ then reads

$$g_{\alpha\beta} = \delta_{ab} e^a_\alpha e^b_\beta \ . \tag{130}$$

It is invariant under the left action of G. Another left-invariant metric is obtained by setting $\gamma_{ab} := C^c_{ad} C^d_{bc}$ to the Killing metric restricted to G/H

$$g_{\alpha\beta} = \gamma_{ab} V^a_\alpha V^b_\beta \ . \tag{131}$$

This metric also fulfills (120) as can be shown using the Jacobi identities of the algebra (114). This implies that the construction of the dual 1-form basis was independent of the choice of representative. Both left-invariant coset-metrics are well-defined because replacing L_y by $L_y h$ is equal to a $SO(n)$-rotation of the vielbein.

Combination of the ansatz for the reduced metric (129) with the analysis in chapter 3 allows the conclusion that coset space reduction of fields in the above manner is guaranteed to be consistent if the unimodularity condition is met.

4.3 Reduced diffeomorphisms algebra

In general, people carrying out dimensional reductions of diffeomorphism-invariant theories would like to keep all field components in the lower-dimensional theory and not impose additional constraints. The reason is that the fields set to zero to achieve consistency transform as gauge fields of the isometry group of the compactification space. One can see this by considering which elements of the gauge group of the higher dimensional theory survive the truncation. The elements of the diffeomorphism group that survive the truncation are those that map quantities satisfying the Killing condition to quantities satisfying the Killing condition. Namely, they need to preserve the foliation defined by the Lie algebra spanned by the Killing vector fields. This subsection studies the gauge subgroup surviving the truncation by considering an arbitrary 1-form Ω expressed in the mixed basis

$$\Omega = \Omega_\mu(x)\mathrm{d}x^\mu + \Omega_a(x)\omega^a \ . \tag{132}$$

It is valid for arbitrary fields, however. A general diffeomorphism is infinitesimally generated by the vector field

$$v = \epsilon^\mu(x,y)\partial_\mu + \rho^a(x,y)\mathbf{Y}_a =: \epsilon + \rho \tag{133}$$

The reduced diffeomorphisms are those that produce variations $\delta\Omega$ such that

$$\delta\Omega = (\delta\Omega_\mu)(x)\mathrm{d}x^\mu + (\delta\Omega_a)(x)\omega^a \tag{134}$$

when expressed in the old, untransformed mixed basis. Note that the crucial point is that the reduced diffeomorphisms must not introduce y-dependencies. A series of short calculations given in Pons and Talavera (2003, p.14ff) shows that the diffeomorphisms belonging to the reduced gauge group are generated by

$$\vec{v} = \epsilon^\mu(x)\partial_\mu + \eta^a(x)\mathbf{Y}_a + \psi^a(y)\mathbf{Y}_a \tag{135}$$

where $e^\mu\partial_\mu$ generated diffeomorphisms in the lower dimensional theory. $\eta^a(x)\mathbf{Y}_a$ generates Yang-Mills transformations and $\psi^a(y)\mathbf{Y}_a$ generate residual rigid symmetries. One can now evaluate how the variation $\delta_{YM} := \eta^a(x)\mathbf{Y}_a$ acts on the components of metric tensor:

$$\begin{aligned}
\delta_{YM} g_{\mu\nu} &= 0 \\
\delta_{YM} g_{ab} &= \eta^d(C^c_{da} g_{cb} + C^c_{db} g_{ac}) \\
\delta_{YM} A^a_\mu &= \partial_\mu \eta^a + A^c_\mu C^a_{cd} \eta^d \\
\delta_{YM} \Omega_y &= \eta^d C^d_{da} \Omega_a \\
\delta_{YM} \Omega_\mu &= 0 \ .
\end{aligned} \tag{136}$$

The metric components g_{ab} transform under the adjoint action of the Yang-Mills gauge group, which means that they are in general charged objects under the Yang Mills transformation. The third equation identifies A^a_μ as the gauge bosons for the Yang-Mills theory associated with the Lie algebra of the Killing vectors. This chapter showed that it is exactly these objects that need to be set to zero in order to obtain consistent coset space reduction in a canonical manner.

5

Miraculous sphere reductions

In coset space reduction, one usually wants to keep the Yang Mills gauge bosons in the lower-dimensional theory. The aim is to build physical models with suitable symmetries by choosing spaces $\mathscr{M} \times M_G$ where $\mathscr{M}$ is the space-time manifold and M_G is a manifold with suitable isometry group G. A common procedure to implement the truncation of the higher-dimensional fields is to perform generalised Fourier expansion in terms of representations of the G. [3] Given a higher-dimensional fields, one obtains what is called a *Kaluza-Klein tower* of infinitely-many lower-dimensional modes. If one were to retain all modes, there is no risk of inconsistency. Yet, unless one discards fields, one just expresses higher-dimensional fields through an infinity of lower-dimensional fields. In the *Kaluza-Klein ansatz*, one wants to discard all but a finite number of states. For instance, one retains the zero eigenvalue modes for the mass operators in the case of the massless particles. It can be shown that these zero eigenvalue modes include the singlet fields under the group action. The threat to consistency is that some of the truncated fields might not remain zero under the transformation of the isometry group G. In other words, the modes retained act as source terms for the modes of the theory to be truncated. As argued in the last chapter, there is no known group-theoretical argument that guarantees consistency if M_G is a coset space and the ansatz for the metric is not invariant under the action of G.

This section demonstrates the appearance of source terms for the truncated fields in the truncated e.o.m. if one makes a metric ansatz retaining the gauge fields for general coset space reductions. Further, the potential for restoring the consistency of such reductions is briefly studied. Mind that consistency can only be restored in the weaker sense: the functional dependence on the higher-dimensional space cancels out when substituting the ansatz into the higher-

[3] See Hinterbichler, Levin and Zukowski (2014) for a reference on Kaluza-Klein towers on general manifolds.

dimensional e.o.m.. This means that all terms acting as *sources* for truncated modes need to cancel out in the reduced higher-dimensional e.o.m.. The above type of inconsistency was first pointed out by Duff, Nilsson, Pope and Warner (1984). The following analysis draws on lecture notes on Kaluza-Klein-theory by Pope (p. 55ff).

5.1 Source terms

Consider a Kaluza-Klein reduction on a manifold $d+n$-dimensional manifold $\mathscr{M} \times M_G$ where M_G has the isometry group G. An ansatz for the metric suppressing the scalar fields is given by

$$d\hat{s}^2 = ds^2 + g_{ab}(y)(dy^a + K^{aI}(y)A^I(x))(dy^b + K^{bJ}(y)A^J(x)) \ . \tag{137}$$

One can read of the components $\hat{g}_{AB}$ of the higher-dimensional metric:

$$\begin{aligned} \hat{g}_{\mu\nu}(x,y) &= g_{\mu\nu}(x) + A^I_\mu(x)A^J_\nu(x)K^{aI}(y)K^{bJ}(y)g_{ab}(y) \\ \hat{g}_{\mu j}(x,y) &= A^I_\mu(x)K^{aI}(y)g_{aj}(y) \\ \hat{g}_{ab}(x,y) &= g_{ab}(y) \end{aligned} \tag{138}$$

where the coordinates x^μ $(\mu,\nu,\eta = 1,...,d)$ refer to $\mathscr{M}$, y^a $(a,b,c,... = d+1,\ ...,\ d+n)$ to M_G and $g_{ab}(y)$ is the metric on M_G. $K^{aI}(y)$ are the Killing vectors corresponding to the isometries of this metric, I runs over the dimension of the isometry group G. Further define $K^I_a := g_{ab}K^{bI}$. The claim is that this is the correct ansatz since substitution into the higher-dimensional action and integration over y, yields the four-dimensional Einstein-Yang-Mills action with metric $g_{\mu\nu}(x)$ and gauge potential $A^I_\mu(x)$. The scalar fields in the above ansatz are ignored which is an obvious source of inconsistency. Yet, this inconsistency can be remedied by reintroducing the scalar fields through a suitable Weyl rescaling (Duff et al., 1984, p. 91f). This section focuses on the inconsistency arising from not setting the gauge potential A^I_μ to zero. As seen earlier, the above metric ansatz will in general not be invariant under the action of the isometry group. Palais' principle does not guarantee that solutions to the lower-dimensional e.o.m. are also solutions to the higher dimensional e.o.m.

Historically, Pauli first used such an ansatz in 1953 to generalise the original Kaluza-Klein theory to a six-dimensional space for obtaining non-abelian gauge symmetry.[4] Pauli arrived at the essentials of an $SU(2)$ gauge theory but discarded and has never published the theory because he saw that one would obtain

[4] See Straumann (2002) for an account of Pauli's work.

vector mesons with rest-mass zero which he considered nonphysical. Interestingly, he also recognised that there was no justification for substituting his ansatz into the higher-dimensional action. Even though he formulated his calculation differently, Pauli essentially considered a six-dimensional total space $\mathscr{M} \times \mathbb{S}^2$ with the isometry group $G = SO(3)$ acting on $\mathbb{S}^2$ in a canonical manner

$$(x, y) \mapsto [x, R(x) \cdot y] \tag{139}$$

In this setting, the quantities in (137) correspond to a Lie-algebra valued 1-form $A = A^I T_I$ with $A^I = A^I_\mu dx^\mu$, the standard generators T_I $(I = 1, 2, 3)$ of the Lie algebra of $SO(3)$ and $K^I := K^{Ia} \frac{\partial}{\partial y^a}$ being the three Killing fields on $\mathbb{S}^2$. g_{mn} is the standard metric on $\mathbb{S}^2$. Omitting the scalar fields, Pauli formulated the ansatz

$$\hat{g} = g - \gamma_{ab}[dy^a + K^{aI} A^I] \otimes [dy^b + K^{bJ} A^J] \ . \tag{140}$$

where $A^I := A^I_\mu(x) dx^\mu$ are the Yang-Mills gauge bosons in the lower-dimensional theory. This section demonstrates that such an ansatz will in general yield an inconsistent truncation by calculating the Ricci-tensor of the higher-dimensional metric. One starts with a convenient choice for the vielbein:

$$\hat{e}^\mu = e^\mu, \qquad \hat{e}^a = e^a + K^{aI} A^I \tag{141}$$

where e^μ is a vielbein for the lower-dimensional metric ds^2 and e^a is a vielbein for the metric on the internal manifold M_G such that $g_{ab} = e^m_a e^m_b$ and where $K^{aI} := e^a_m K^{mI}$ was defined. The full calculations leading to the vielbein components of the Ricci tensors $R_{\mu\nu}$ and R_{ab} for the lower-dimensional metric $g_{\mu\nu}$ and g_{ab} can be found in appendix A.1. They are given as

$$\begin{aligned} \hat{R}_{\mu\nu} =& R_{\mu\nu} - \frac{1}{2} K^{aI} K^J_a F^{I\lambda}_\mu F^J_{\lambda\nu} \\ \hat{R}_{ab} =& R_{ab} + \frac{1}{4} K^I_a K^J_b F^I_{\eta\rho} F^{J\eta\rho} \ . \end{aligned} \tag{142}$$

The Ricci scalar $\hat{R}$ is given as

$$\begin{aligned} \hat{R} =& \hat{R}^A_A = \hat{R}^\mu_\mu + \hat{R}^a_a \\ =& R^a_a + R^\mu_\mu + \frac{1}{4} K^{aI} K^J_a F^I_{\eta\rho} F^{J\eta\rho} \\ & - \frac{1}{2} K^{aI} K^J_a F^I_{\eta\rho} F^{J\eta\rho} \\ & = R - \frac{1}{4} K^{Ia} K^J_a F^I_{\eta\rho} F^{J\eta\rho} \ . \end{aligned} \tag{143}$$

Now consider pure gravity as example of a higher-dimensional theory with the equations of motion being $\hat{R}_{AB} = 0$. The inconsistency occurs in the lower-dimensional Einstein equation with the Yang-Mills fields acting as a source:

$$\begin{aligned} 0 &= \hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}\eta_{\mu\nu} \\ \iff R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} &= \frac{1}{2}K^{aI}K_a^J[F^I_{\mu\rho}F^{J\rho}_{\nu} - \frac{1}{4}F^I_{\rho\sigma}F^{J\rho\sigma}\eta_{\mu\nu}] \; . \end{aligned} \tag{144}$$

Note that the indices I and J range over dim G values whereas a ranges from 1 to dim M_G. Thus, if G and M_G have different dimensions, $K^{aI}K_a^J$ is necessarily a degenerate matrix and cannot possibly be proportional to the identity matrix. Furthermore, $K^{aI}K_a^J$ term depends on the coordinates y^m of the internal space whereas the other terms are y-independent because of the reduction ansatz. Thus, one obtains a y-dependence in the lower dimensional theory. For instance, if one choose $M_G = S^k$ with its $SO(k+1)$ invariant metric then $M_G{=}SO(k+1)/SO(k)$ and the Killing vectors do not satisfy

$$K^{aI}(y)K_a^J(y) = \delta^{IJ} \tag{145}$$

but

$$K^{aI}(y)K_a^J(y) = \delta^{IJ} + Y^{IJ}(y) \tag{146}$$

where $Y^{IJ}(y)$ is a harmonic of the scalar Laplacian belonging to the representation of $SO(k+1)$. In other words, $K^{aI}(y)K_a^J$ acts as a *source* for terms that were truncated in the reduction process. Potential remedies to this issues include selecting a subgroup $G' \subseteq G$ with Killing vectors $K_a^{I'}$ where I' runs of the dimension of G' such that

$$K^{aI'}(y)K_a^{J'}(y) = \delta^{I'J'} \; . \tag{147}$$

But this implies the existence of everywhere non-vanishing vector fields which is a condition that cannot be satisfied on spaces with non-zero Euler number ξ, for instance. This includes spheres of even dimension and the complex projective space of any dimension. In general, one will only obtain $K^{aI}(y)K_a^J(y) = \delta^{IJ}$ for a dimensional reduction on group manifolds. The dimensional reduction on coset spaces with the above ansatz is therefore inconsistent.

There are only a few known examples for consistent coset space reductions on spheres including the reduction of $d = 11$ supergravity to $d = 4$ gauged $N = 8$ supergravity on the coset space $\mathbb{S}^7$ (Wit & Nicolai, 1987); the reduction of $d = 11$ supergravity to $d = 7$ gauged $N = 2$ supergravity on the coset space $\mathbb{S}^4$ (Nastase, Vaman & van Nieuwenhuizen, 1999, 2000); the reduction of $d = 10$ type IIB supergravity to $d = 5$ gauged $N = 8$ supergravity on $\mathbb{S}^5$ (Cvetic, Lu, Pope, Sadrzadeh & Tran, 2000). These have become known as *miracles* since for many years there was no satisfying theoretical explanation for their consistency.

This chapter concludes by briefly outlining how consistency in the weak sense is restored in these *miraculous sphere reductions* at the example of the reduction of 11-dimensional gravity on $\mathbb{S}^4$. The starting point of such a dimensional reduction is the bosonic Lagrangian for eleven-dimensional supergravity

$$\mathcal{L}_{11} = \hat{R}\hat{*}\mathbb{I} - \frac{1}{2}\hat{F}_{(4)} \wedge \hat{F}_{(4)} + \frac{1}{6}\hat{F}_{(4)} \wedge \hat{F}_{(4)} \wedge \hat{A}_{(3)} \ . \tag{148}$$

The reduction ansatz is chosen as in the preceding section. The Killing vectors of $\mathbb{S}^4$ can be shown to satisfy

$$K^{aI}K^J_a + \frac{1}{2}g^{-2}L^{abI}L^J_{ab} = \delta_{IJ} \tag{149}$$

after appropriate renormalisation and where $L^I_{(2)} = dK^I$ with $K^I := K^I_a e^a$. The lower dimensional field equations turn out to be

$$R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} = \frac{1}{2}Y^{IJ}[F^I_{\mu\rho}F^{J\rho}_\nu - \frac{1}{4}F^I_{\rho\sigma}F^{J\rho\sigma}\eta_{\mu\nu}] - \frac{15}{4}g^2\eta_{\mu\nu} \tag{150}$$

with $Y^{IJ} = K^{aI}K^J_a + \frac{1}{2}g^{-2}L^{abI}L^J_{ab}$. The potential obstruction to the consistency of the truncation pointed out in this section does not occur because the additional terms coming from the supersymmetric Lagrangian exactly cancel out the coordinate-dependence on the internal space introduced by the source term $K^{aI}(y)K^J_a(y)$.

6

Conclusions and Outlook

The thesis clarified the issue of *consistent Kaluza-Klein dimensional reduction* in two parts. In Kaluza-Klein dimensional reduction, one reduces the dimension of the space-time underlying a variational principle by making use of symmetries. However, the solutions of the reduced variational principle are not always solutions of the original variational principle and the dimensional reduction may be inconsistent. In the first part, the thesis gave the main ingredients of the proof of the Palais' theorem guaranteeing consistent truncation when restricting the Lagrangian to the set of symmetric points under the action of a *compact Lie group*. It was found that the issue of consistent truncation can be decided on arbitrary Banach manifolds with simple algebraic and differential geometric arguments. It was further explained how the *Principle of Symmetric Criticality* relates to the proof of the *unimodularity condition* on Lorentz manifolds. The unimodularity condition guarantees consistent truncation under a Lie algebra of independent Killing vector fields ξ^a. One expresses a higher-dimensional gauge field A of a group manifold G as

$$A = A_\mu dx^\mu + A_a(x)e^a(y) \tag{151}$$

where e^a are the left-invariant 1-forms under the Lie group action. One keeps the G_L singlets under the full isometry group $G_L \times G_r$ and can describe the truncation with the invariance condition

$$\mathcal{L}_{\xi^a} A = 0 \ . \tag{152}$$

Implementation of the truncation at the level of the Lagrangian and at the level of the e.o.m. shows that the non-commutativity between the two procedures stems from the *integration by parts* required for obtaining the Lagrange equations. If the isometry group G on the internal space is well-behaved, i.e. compact without boundary, no issue occurs. Yet, if G is non-unimodular, which also implies that G is non-compact, one cannot assume that G-invariant variations vanish outside a compact set or at infinity. A further perspective on the issue is given in Torre (2010). They consider a local field theoretic version of the Principle of Symmetric Criticality and give a generalisation of the unimodularity condition by formulating the variation in terms of differential forms.

In the second part, this thesis showed that in implementing an a-priori consistent truncation on coset spaces, one sacrifices the Yang Mills gauge bosons A^b_ν as well as the fields g_{ab} which acts as charged objects for the Yang-Mills-field. This is because, one needs to enforce

$$C^c_{ia} g_{cb} + C^c_{ib} g_{ac} = 0 \qquad A^b_\nu = 0 \tag{153}$$

in order to render the metric G-invariant. In the case of independent Killing vectors all components of the metric (or an arbitrary field) survive the truncation. Note that in general, one has to control the consequences of the truncation of the variational principle through the imposition of constraints. It turns out the conditions for consistency of such truncations can be understood using Dirac-Bergmann's theory of constraint systems (Anderson & Bergmann, 1951). Pons and Talavera (2004) provide a good starting point for a further look into the issue.

When carrying out coset space reductions, one usually wants to keep the Yang-Mills gauge bosons in the lower-dimensional theory by making an ansatz for the higher-dimensional fields that is not G-invariant. One then obtains a $SO(n)$ as symmetry group operating in the lower dimensional theory in the case of spherical reductions, for instance. However, there are only known examples where such a truncation turned out to be consistent without demanding additional conditions. These have become known in the literature as *miracles* because for many years there was no known group-theoretical explanation as to why these truncations were consistent. Recently, there have been several advances in the explanation of the consistency of Kaluza-Klein reductions using the tools of *generalised geometry and exceptional generalised geometries (EEG)* (Inverso, 2017; Lee, Strickland-Constable & Waldram, 2017; Cassani, Josse, Petrini & Waldram, 2019). Lee et al. (2017) in particular explain the consistency of spherical reductions using the concept of *generalised parallelisability.* Further research could be conducted on whether their results can also be understood in the context of Palais' principle.

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A

Appendix

A.1 Ricci tensor in a general coset space reduction

Consider a Kaluza-Klein reduction on $d+n$-dimensional manifold $\mathscr{M} \times M_G$ where M_G has the isometry group G. This section aims to calculate the Ricci tensor from the following ansatz for the metric on $\mathscr{M} \times M_G$:

$$\begin{aligned} \hat{g}_{\mu\nu}(x,y) &= g_{\mu\nu}(x) + A^I_\mu(x) A^J_\nu(x) K^{aI}(y) K^{bJ}(y) g_{ab}(y) \\ \hat{g}_{\mu j}(x,y) &= A^I_\mu(x) K^{aI}(y) g_{aj}(y) \\ \hat{g}_{ab}(x,y) &= g_{ab}(y) \end{aligned} \tag{154}$$

with respect to the vielbein

$$\hat{e}^\mu = e^\mu; \qquad \hat{e}^a = e^a + K^{aI} A^I \tag{155}$$

where e^μ is a vielbein for the lower-dimensional metric ds^2 and e^a is a vielbein for the metric on the internal manifold M_G such that $g_{ab} = e^m_a e^m_b$ and where $K^{aI} := e^a_m K^{mI}$ was defined. The changes of basis are carried out using the following relations:

$$\begin{aligned} e^a &= e^a_m dy^m \\ dy^m &= e^m_a e^a \\ e^\mu &= E^\mu_\nu dx^\nu \\ dx^\nu &= (E^{-1})^\nu_\mu e^\mu \end{aligned} \tag{156}$$

In the first step, one needs to calculate the exterior derivatives of the higher- and lower-dimensional vielbein basis vectors:

$$
\begin{aligned}
d\hat{e}^{\mu} &= de^{\mu} \\
d\hat{e}^{a} &= de^{a} + d(K^{aI}A^{I}) \\
&= de^{a} + \partial_b K^{aI} A^{I}_{\nu} (E^{-1})^{\nu}_{\mu} e^{b} \wedge \hat{e}^{\mu} + K^{aI} \partial_{x^{\nu}} A^{I}_{\mu} (E^{-1})^{\nu}_{\eta} (E^{-1})^{\mu}_{\rho} \hat{e}^{\eta} \wedge \hat{e}^{\rho} \\
&= de^{a} + \partial_b K^{aI} A^{I}_{\nu} (E^{-1})^{\nu}_{\mu} e^{b} \wedge \hat{e}^{\mu} + K^{aI} \partial_{x^{\nu}} A^{I}_{\mu} (E^{-1})^{\nu}_{\eta} (E^{-1})^{\mu}_{\rho} \hat{e}^{\eta} \wedge \hat{e}^{\rho} \\
&= de^{a} + \partial_b K^{aI} A^{I}_{\eta} \hat{e}^{b} \wedge \hat{e}^{\eta} + (\partial_b K^{aI} K^{bJ} A^{I}_{\eta} A^{J}_{\rho} + \frac{1}{2} K^{aI} f^{I}_{\eta\rho}) \hat{e}^{\eta} \wedge \hat{e}^{\rho}
\end{aligned}
\tag{157}
$$

where $\partial_a := e^m_a \frac{\partial}{\partial y^m}$ was introduced and all components in the last expression refer to vielbein indices. Furthermore, $f^I_{\mu\nu}$ denotes the components of the field strength $f^I = dA^I$ with respect to the vielbein $\hat{e}^{\mu}$:

$$
f^I_{\mu\nu} := \partial_\mu A^I_\nu - \partial_\nu A^I_\mu \ . \tag{158}
$$

Using vanishing torsion, Cartan's first structure equation yields

$$
\begin{aligned}
0 &= d\hat{e}^{\mu} + \hat{\omega}^{\mu}_{\nu} \wedge \hat{e}^{\nu} + \hat{\omega}^{\mu}_{a} \wedge \hat{e}^{a} \\
&= \ - \omega^{\mu}_{\nu} \wedge \hat{e}^{\nu} + \hat{\omega}^{\mu}_{\nu} \wedge \hat{e}^{\nu} + \hat{\omega}^{\mu}_{a} \wedge \hat{e}^{a} \\
0 &= d\hat{e}^{a} + \hat{\omega}^{a}_{b} \wedge \hat{e}^{b} + \hat{\omega}^{a}_{\nu} \wedge \hat{e}^{\nu} \\
&= + de^{a} + \hat{\omega}^{a}_{b} \wedge \hat{e}^{b} + \hat{\omega}^{a}_{\nu} \wedge \hat{e}^{\nu} \\
&\quad + \partial_b K^{aI} A^{I}_{\eta} \hat{e}^{b} \wedge \hat{e}^{\eta} \\
&\quad + \partial_b K^{aI} K^{bJ} A^{I}_{\eta} A^{J}_{\rho} \hat{e}^{\eta} \wedge \hat{e}^{\rho} \\
&\quad + \frac{1}{2} K^{aI} f^{I}_{\eta\rho} \hat{e}^{\eta} \wedge \hat{e}^{\rho} \\
&= - \omega^{a}_{b} \wedge \hat{e}^{b} + \hat{\omega}^{a}_{b} \wedge \hat{e}^{b} + \hat{\omega}^{a}_{\eta} \wedge \hat{e}^{\eta} \\
&\quad + K^{bJ} A^{J}_{\rho} \omega^{a}_{b} \wedge \hat{e}^{\eta} \\
&\quad - \partial_b K^{aI} A^{I}_{\eta} \hat{e}^{\eta} \wedge \hat{e}^{b} \\
&\quad + \partial_b K^{aI} K^{bJ} A^{I}_{\eta} A^{J}_{\rho} \hat{e}^{\eta} \wedge \hat{e}^{\rho} \\
&\quad + \frac{1}{2} K^{aI} f^{I}_{\eta\rho} \hat{e}^{\eta} \wedge \hat{e}^{\rho}
\end{aligned}
\tag{159}
$$

where Cartan's first structure equation for the vielbeins on $\mathscr{M}$ and M_G were used. Note that $\hat{\omega}_{AB} = -\hat{\omega}_{BA}$ because the metric is constant with respect to the vielbein. Defining $F^I_{\mu\nu}$ which denotes components of the Yang-Mills field strength with respect to the vielbein $\hat{e}^\mu$:

$$F^I_{\mu\nu} := f^I_{\mu\nu} + C^I_{KJ} A^K_\mu A^J_\nu \tag{160}$$

with C^K_{IJ} being the structure constants of the Lie algebra satisfied by the Killing vectors, one obtains

$$\begin{aligned}
\hat{\omega}_{\mu\nu} &= \omega_{\mu\nu} - \frac{1}{2} K^{aI} F^I_{\mu\nu} \hat{e}^a \\
\hat{\omega}_{\mu a} &= -\frac{1}{2} K^{aI} F^I_{\mu\nu} \hat{e}^\nu \\
\hat{\omega}_{ab} &= \omega_{ab} + \nabla_a K^I_b A^I_\mu \hat{e}^\mu
\end{aligned} \tag{161}$$

where $\omega^{\mu\nu}$ is the spin connection for the lower-dimensional vielbein e^μ and ω^{ab} is the spin connection for the vielbein e^a on the internal manifold M_G. Note that the partial derivative in the last line could be exchanged with a covariant derivative because of the antisymmetry between a and b and $\nabla_a K^I_b = -\nabla_b K^I_a$ from the *Killing equation*. In the next step, one calculates the exterior derivatives of the connection 1-forms:

$$\begin{aligned}
d\hat{\omega}_{\mu\nu} =& + d\omega_{\mu\nu} \\
& - \frac{1}{2} K^{aI} F^I_{\mu\nu} d\hat{e}^a \\
& - \frac{1}{2} \nabla_b K^{aI} F^I_{\mu\nu} \hat{e}^b \wedge \hat{e}^a \\
& + \frac{1}{2} \partial_b K^{aI} F^I_{\mu\nu} K^{bJ} A^J_\eta \hat{e}^\eta \wedge \hat{e}^a \\
& - \frac{1}{2} K^{aI} \partial_\eta F^I_{\mu\nu} \hat{e}^\eta \wedge \hat{e}^a \\
d\hat{\omega}_{\mu a} =& + \frac{1}{2} \partial_b K^{aI} F^I_{\mu\eta} \hat{e}^\eta \wedge \hat{e}^b \\
& + \frac{1}{2} \partial_b K^{aI} F^I_{\mu\rho} K^{bJ} A^J_\eta \hat{e}^\eta \wedge \hat{e}^\rho \\
& - \frac{1}{2} K^{aI} \partial_\eta F^I_{\mu\rho} \hat{e}^\eta \wedge \hat{e}^\rho \\
d\hat{\omega}_{ab} =& + d\omega_{ab} \\
& + \partial_c \nabla_a K^I_b A^I_\mu \hat{e}^c \wedge \hat{e}^\mu \\
& + \frac{1}{2} \nabla_a K^I_b f^I_{\eta\rho} \hat{e}^\eta \wedge \hat{e}^\rho \\
& + \partial_c \nabla_a K^I_b K^{cJ} A^I_\eta A^J_\rho \hat{e}^\eta \wedge \hat{e}^\rho \ .
\end{aligned} \tag{162}$$

Cartan's second structure equation then yields:

$$\begin{aligned}
\hat{\Omega}^{\mu}_{\nu} &= + d\hat{\omega}^{\mu}_{\nu} + \hat{\omega}^{\mu}_{\eta} \wedge \hat{\omega}^{\eta}_{\nu} + \hat{\omega}^{\mu}_{a} \wedge \hat{\omega}^{a}_{\nu} \\
&= + \Omega^{\mu}_{\nu} \\
&\quad + \frac{1}{2}\partial_b K^{aI} F^{I\mu}_{\nu} \hat{e}^a \wedge \hat{e}^b \\
&\quad + \frac{1}{4} K^{aI} K^{bJ} F^{I\mu}_{\eta} F^{J\eta}_{\nu} \hat{e}^a \wedge \hat{e}^b \\
&\quad + \frac{1}{2} K^{aI} F^{I\mu}_{\nu} \omega_{ab} \wedge \hat{e}^b \\
&\quad - \frac{1}{2} K^{aI} F^{I\eta}_{\nu} \omega^{\mu}_{\eta} \wedge \hat{e}^a \\
&\quad + \frac{1}{2} K^{aI} F^{I\mu}_{\eta} \omega^{\eta}_{\nu} \wedge \hat{e}^a \\
&\quad - \frac{1}{2} K^{bI} \partial_\eta F^{I\mu}_{\nu} \hat{e}^\eta \wedge \hat{e}^b \\
&\quad + K^{aI} F^{I\mu}_{\nu} \partial_a K^J_b A^J_\eta \hat{e}^\eta \wedge \hat{e}^b \\
&\quad - \frac{1}{4} K^{aI} K^J_a F^I_{\mu\nu} F^J_{\eta\rho} \hat{e}^\eta \wedge \hat{e}^\rho \\
&\quad - \frac{1}{4} K^{aI} K^J_a F^I_{\mu\eta} F^J_{\nu\rho} \hat{e}^\eta \wedge \hat{e}^\rho
\end{aligned} \tag{163}$$

$$\begin{aligned}
\hat{\Omega}^{\mu}_{a} &= + d\hat{\omega}^{\mu}_{a} + \hat{\omega}^{\mu}_{\sigma} \wedge \hat{\omega}^{\sigma}_{a} + \hat{\omega}^{\mu}_{b} \wedge \hat{\omega}^{b}_{a} \\
&= - \frac{1}{2} K^I_a F^{I\sigma}_{\eta} \omega^{\mu}_{\sigma} \wedge \hat{e}^\eta \\
&\quad - \frac{1}{2} K^I_b F^{I\mu}_{\eta} \hat{e}^\eta \wedge \omega^b_a \\
&\quad + \frac{1}{2} \partial_b K^I_a F^{I\mu}_{\eta} \hat{e}^\eta \wedge \hat{e}^b \\
&\quad - \frac{1}{4} K^J_b K^I_a F^{J\mu}_{\sigma} F^{I\sigma}_{\eta} \hat{e}^\eta \wedge \hat{e}^b \\
&\quad + \frac{1}{2} \partial_b K^I_a F^{I\mu}_{\rho} K^{bJ} A^J_\eta \hat{e}^\eta \wedge \hat{e}^\rho \\
&\quad + \frac{1}{2} K^I_b F^{I\mu}_{\eta} \nabla_a K^{bJ} A^J_\rho \hat{e}^\eta \wedge \hat{e}^\rho \\
&\quad - \frac{1}{2} K^I_a \partial_\eta F^{I\mu}_{\rho} \hat{e}^\eta \wedge \hat{e}^\rho
\end{aligned} \tag{164}$$

$$
\begin{aligned}
\hat{\Omega}^a_b =& + d\hat{\omega}^a_b + \hat{\omega}^a_\sigma \wedge \hat{\omega}^\sigma_b + \hat{\omega}^a_c \wedge \hat{\omega}^c_b \\
=& + \Omega^a_b \\
& + \nabla_c K^I_b A^I_\eta \omega^a_c \wedge \hat{e}^\eta \\
& + \nabla_a K^I_c A^I_\eta \hat{e}^\eta \wedge \omega^c_b \\
& - \partial_c \nabla_a K^I_b A^I_\eta \hat{e}^\eta \wedge \hat{e}^c \\
& + \frac{1}{2} \nabla_a K^I_b f^I_{\eta\rho} \hat{e}^\eta \wedge \hat{e}^\rho \\
& + \partial_c \nabla_a K^I_b K^{cJ} A^I_\eta A^J_\rho \hat{e}^\eta \wedge \hat{e}^\rho \\
& + \frac{1}{4} K^{aI} K^J_b F^{J\sigma}_\rho F^I_{\sigma\eta} \hat{e}^\eta \wedge \hat{e}^\rho \\
& + \nabla_a K^{cI} \nabla_c K^J_b A^I_\eta A^J_\rho \hat{e}^\eta \wedge \hat{e}^\rho
\end{aligned}
\tag{165}
$$

where Ω^μ_ν and Ω^a_b are the curvature 2-forms for the lower-dimensional metric $g_{\mu\nu}$ and the internal metric g_{ab} respectively. Note that whenever Greek indices in the Yang-Mills field strength were raised to make clear which quantities were summed up, it was the left index which was raised. The components of the Ricci tensor now follow as the trace over the components of the curvature 2-forms:

$$
\begin{aligned}
\hat{R}_{\mu\nu} =& \hat{R}^A_{\mu A\nu} = \hat{\Omega}^\lambda_\mu(\hat{e}_\lambda, \hat{e}_\nu) + \hat{\Omega}^a_\mu(\hat{e}_a, \hat{e}_\nu) \\
=& R_{\mu\nu} \\
& - \frac{1}{4} K^{aI} K^J_a \left(F^{I\lambda}_\mu F^J_{\eta\rho} + F^{I\lambda}_\eta F^J_{\mu\rho} \right) \left(\delta^\eta_\lambda \delta^\rho_\nu - \delta^\eta_\nu \delta^\rho_\lambda \right) \\
& + \frac{1}{4} K^J_b K^{aI} F^J_{\mu\lambda} F^{I\lambda}_\eta (-\delta^b_a \delta^\eta_\nu) \\
=& R_{\mu\nu} - \frac{1}{2} K^{aI} K^J_a F^{I\lambda}_\mu F^J_{\lambda\nu}
\end{aligned}
\tag{166}
$$

where it was used that the divergence of Killing vectors vanishes and where $R_{\mu\nu}$ is the Ricci tensors for the lower-dimensional metric $g_{\mu\nu}$.

$$
\begin{aligned}
\hat{R}_{ab} &= \hat{R}^A_{aAb} = \hat{\Omega}^\lambda_a(\hat{e}_\lambda, \hat{e}_b) + \hat{\Omega}^c_a(\hat{e}_c, \hat{e}_b) \\
&= R_{ab} + \frac{1}{4} K^I_a K^J_b F^I_{\eta\rho} F^{J\eta\rho}
\end{aligned}
\tag{167}
$$

where the second term in the first line contributed the Ricci tensor R_{ab} of the metric g_{ab} on the higher-dimensional manifold.

Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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